Relaxation Methods Applied to Engineering Problems. VIIIA. Problems Relating to Large Transverse Displacements of Thin Elastic Plates

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RELAXATION METHODS APPLIED TO ENGINEERING PROBLEMS

VIII A. PROBLEMS RELATING TO LARGE TRANSVERSE DISPLACEMENTS OF THIN ELASTIC PLATES

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Small transverse displacements of a flat elastic plate are governed by a single linear equation, but large displacements entail stretching of the middle surface and consequent tensions, which interacting with the curvatures (i.e. by 'membrane effect') introduce non-linear terms into the conditions of equilibrium and so make those equations no longer independent. The second-order terms were formulated by von Kármán in 1910, but the amended ('large deflexion') equations have been solved only in a few cases, and then with considerable difficulty.

In this paper four examples are treated approximately by a technique based on relaxation methods. The first and second are relatively simple problems which have been solved exactly and so serve as test cases, viz. (*a*) a circular plate, with clamped edge, which sustains a uniform transverse pressure and (*b*) a circular plate, with 'simply supported' edge, which buckles with radial symmetry under uniform edge thrust. The third and fourth examples present great difficulties to orthodox analysis: they are (*c*) a square plate, sustaining uniform transverse pressure, of which the edges are clamped, (*d*) a square plate buckled by actions which, clamping its edges, tend *initially* to induce a state of uniform shear.

Introduction

1. The classical theory of flexure for a thin elastic plate relates the transverse deflexion (w) of the middle surface with the surface intensity (Z) of transverse loading by the equation

$$D\nabla^4 w = Z, \tag{1}$$

in which D stands for the 'flexural rigidity' $\dfrac{2}{3}\dfrac{Eh^3}{1-\sigma^2}$ * and

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{2}$$

It is known that the theory has restricted application; for on the one hand its basic assumptions can be questioned unless the plate is thin, and on the other it neglects an effect which must be sensible when w has values comparable with the thickness. This is the 'membrane effect' of curvature, whereby tension or compression in the middle surface tends to oppose or to reinforce Z. It is negligible when w is very small, provided that no stresses act initially in the plane of the middle surface; but even so it operates

* h denotes the half-thickness of the plate, E is Young's modulus, σ is Poisson's ratio. The notation of this paper follows that of Parts VII A and VII B of the series.

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when w is large, because stretching of that surface is a necessary consequence of transverse deflexion. It entails great difficulty in analysis, whether the problem be concerned with equilibrium or with elastic stability.

BASIC THEORY. (1) PROBLEMS OF STATIC EQUILIBRIUM

2. von Kármán (1910) was the first to formulate these statements mathematically. Suppose in the first place that w is *not* accompanied by displacements u, v in the middle surface. Then to a first approximation (i.e. with neglect of terms of the third and higher orders in w and its derivatives) the sides and diagonals of an initially square element $ABCD$ of the middle surface become (cf. figure 1)

$$AB = CD = L\left\{1+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \ldots\right\}, \qquad AC = L\sqrt{2}\left\{1+\frac{1}{4}\left(\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}\right)^2 \ldots\right\},$$

$$AD = BC = L\left\{1+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \ldots\right\}, \qquad BD = L\sqrt{2}\left\{1+\frac{1}{4}\left(\frac{\partial w}{\partial x}-\frac{\partial w}{\partial y}\right)^2 \ldots\right\},$$

so the strains in the middle surface at the point considered are (to the same approximation)

$$e_{xx} = \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \quad e_{yy} = \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2, \quad e_{xy} = \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}.$$

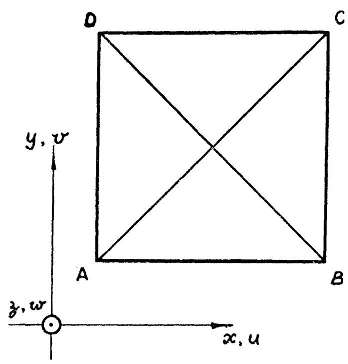

FIGURE 1

The strains due to u and v have their usual expressions, and manifestly can be added. Consequently under conditions of *plane stress* (Z_x, Z_y and Z_z zero everywhere) we have

$$\left.\begin{aligned} \frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 &= e_{xx} = \frac{1}{E}(X_x-\sigma Y_y), \\ \frac{\partial v}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 &= e_{yy} = \frac{1}{E}(Y_y-\sigma X_x), \\ \left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} &= e_{xy} = \frac{2(1+\sigma)}{E}X_y, \end{aligned}\right\} \tag{3}$$

from which u and v may be eliminated to obtain a relation between w and X_x, Y_y, X_y. These last are the *additional* stresses due to large deflexion.

When no force is operative in the direction of x or y, the equations of equilibrium are

$$\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}=0, \quad \frac{\partial X_y}{\partial x}+\frac{\partial Y_y}{\partial y}=0, \tag{4}$$

so we may write

$$X_x=\frac{\partial^2\chi}{\partial y^2}, \quad Y_y=\frac{\partial^2\chi}{\partial x^2}, \quad X_y=-\frac{\partial^2\chi}{\partial x\,\partial y}, \tag{5}$$

where χ is 'Airy's stress function'. Then the result of eliminating u and v is

$$\nabla^4\chi=E\left\{\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right\}, \tag{6}$$

which—in conjunction with boundary conditions derived from (5)—would serve to determine χ if w were known. It reduces to the customary biharmonic equation

$$\nabla^4\chi=0 \tag{7}$$

when terms of the second order in w are neglected.

3. When such neglect is not permissible, equation (1) must be modified to take account of X_x, Y_y, X_y. The corresponding stress resultants are ($2h$ denoting the thickness: cf. footnote to § 1)

$$T_x=2h\,.\,X_x, \quad T_y=2h\,.\,Y_y, \quad S_{xy}=2h\,.\,X_y, \tag{8}$$

and they contribute an effective transverse loading

$$\frac{\partial}{\partial x}\left(T_x\frac{\partial w}{\partial x}\right)+\frac{\partial}{\partial y}\left(T_y\frac{\partial w}{\partial y}\right)+\frac{\partial}{\partial x}\left(S_{xy}\frac{\partial w}{\partial y}\right)+\frac{\partial}{\partial y}\left(S_{xy}\frac{\partial w}{\partial x}\right) \tag{9}$$

to the right-hand side of (1). Having regard to (4) and (5), we deduce that (1) must be replaced by

$$\nabla^4 w=\frac{2h}{D}\left[\frac{Z}{2h}+\frac{\partial^2\chi}{\partial y^2}\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2\chi}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-2\frac{\partial^2\chi}{\partial x\,\partial y}\frac{\partial^2 w}{\partial x\,\partial y}\right], \tag{10}$$

which—in conjunction with the boundary conditions imposed on w—would serve to determine w if χ were known.

4. In fact neither χ nor w is known initially, so both must be deduced from (6) and (10) combined with the imposed boundary conditions. These are von Kármán's equations (§ 2).

When the boundary values of u and v are specified, introduction of χ is less convenient. Instead, by substitution for X_x, Y_y, X_y in (4) from (3), we can derive two equations to replace (6), as under:

$$\left.\begin{aligned}
&\frac{\partial\Delta}{\partial x}+\frac{1-\sigma}{1+\sigma}\nabla^2 u+\frac{1}{2}\frac{\partial}{\partial x}\left\{\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right\}+\frac{1-\sigma}{1+\sigma}\frac{\partial w}{\partial x}\nabla^2 w=0,\\
&\frac{\partial\Delta}{\partial y}+\frac{1-\sigma}{1+\sigma}\nabla^2 v+\frac{1}{2}\frac{\partial}{\partial y}\left\{\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right\}+\frac{1-\sigma}{1+\sigma}\frac{\partial w}{\partial y}\nabla^2 w=0,\\
&\qquad\left(\Delta\equiv\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right).
\end{aligned}\right\} \tag{11}$$

These, in conjunction with the boundary conditions, would serve to determine u and v if w were known. When terms of the second order in w are neglected they reduce to equations (10) of Part VII A, § 6.

Basic theory. (2) Problems of elastic stability

5. Harder problems are presented by large deflexions as these affect the stability of flat plating subjected to edge thrusts. Here, when w is infinitesimal, equation (1) is replaced (Southwell 1941, § 497) by

$$D\nabla^4 w + \frac{\partial}{\partial x}\left(P_x \frac{\partial w}{\partial x} - S\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial y}\left(P_y \frac{\partial w}{\partial y} - S\frac{\partial w}{\partial x}\right) = 0, \tag{12}$$

P_x, P_y denoting the *thrusts* in the directions Ox, Oy, and S the shearing action, in the plane of the middle surface. Equation (12) is a first approximation which neglects the influence of w upon these stress components: on that understanding P_x, P_y, S_{xy} define the stress system which acts initially, so their *relative* intensities are given. Their absolute intensities, with w, are to be regarded as unknowns, determinable from (12) combined with specified boundary conditions of restraint; and these (normally) are such that a solution can be multiplied by an arbitrary factor (P_x, P_y, S being held constant) without violation of any governing condition. But transverse deflexion w will, as in § 2, entail extension and shear of the middle surface, consequently stresses which will modify the initial stress system P_x, P_y, S_{xy} so as to make (on the whole) for tension and thereby for stability. A more complete treatment may thus be expected to indicate that w cannot in fact be so multiplied: higher initial stresses will be required to maintain deflexions of greater magnitude, and the form of w will therefore alter as its amplitude is increased.

6. The extensional stresses which result from large deflexions (w) can be deduced in the manner of §§ 2–3. When edge tractions are specified, they may be expressed by (5) in terms of a function χ which is governed by (6): when u and v are specified on the boundary, these may be calculated from (11), and from them second derivatives of χ may be deduced in accordance with (3) and (5). The governing equation (12) is now replaced by

$$D\nabla^4 w + \frac{\partial}{\partial x}\left(P_x \frac{\partial w}{\partial x} - S\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial y}\left(P_y \frac{\partial w}{\partial y} - S\frac{\partial w}{\partial x}\right)$$
$$= 2h\left[\frac{\partial}{\partial x}\left(\frac{\partial^2\chi}{\partial y^2}\frac{\partial w}{\partial x} - \frac{\partial^2\chi}{\partial x\,\partial y}\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial^2\chi}{\partial x^2}\frac{\partial w}{\partial y} - \frac{\partial^2\chi}{\partial x\,\partial y}\frac{\partial w}{\partial x}\right)\right], \tag{13}$$

Z being zero and P_x, P_y, S denoting *initial* stresses as in § 5. Because P_x, P_y, S satisfy the equations

$$-\frac{\partial}{\partial x}P_x + \frac{\partial}{\partial y}S = 0, \quad \frac{\partial}{\partial x}S - \frac{\partial}{\partial y}P_y = 0, \tag{14}$$

a form equivalent to (13) is

$$D\nabla^4 w = \frac{\partial^2 w}{\partial x^2}\left(2h\frac{\partial^2 \chi}{\partial y^2} - P_x\right) + \frac{\partial^2 w}{\partial y^2}\left(2h\frac{\partial^2 \chi}{\partial x^2} - P_y\right) - 2\frac{\partial^2 w}{\partial x\,\partial y}\left(2h\frac{\partial^2 \chi}{\partial x\,\partial y} - S\right). \tag{15}$$

We have to solve (13) or (15) in conjunction with (6) or (11), thereby determining both the mode of distortion (w) and the intensity of the action (P_x, P_y, S) which is required to maintain it.

Relaxation methods applied to simple examples.

(1) An example of static equilibrium

7. Our treatment of the problems thus presented is best explained in relation to simple examples. Both classes are exemplified by a circular plate which distorts into a solid of revolution, first on account of transverse pressure and secondly on account of edge thrust acting in its plane; and for both the computations are specially simple in that only one independent variable (r) enters into the governing equations.

Way (1934) has dealt at length with a circular plate, uniformly loaded, which at its edge ($r = a$) is clamped and also constrained against radial displacement resulting from tensions in the middle surface. This will serve to illustrate §§ 2–4.

Replacing Cartesian by polar co-ordinates, and suppressing (for symmetry) all differentials with respect to θ in (6), (7) and (10), we may substitute

$$\left.\begin{aligned} &\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\left(= \frac{1}{r}\frac{d}{dr}r\frac{d}{dr}\right) \text{ for } \nabla^2, \quad \frac{d^2}{dr^2} \text{ for } \frac{\partial^2}{\partial x^2},\\ &\frac{1}{r}\frac{d}{dr} \text{ for } \frac{\partial^2}{\partial y^2}, \quad 0 \text{ for } \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial \theta}\right) = \frac{\partial^2}{\partial x\,\partial y}.\end{aligned}\right\} \tag{16}$$

Then (6) takes the form

$$\nabla^4 \chi + \frac{E}{r}\frac{dw}{dr}\frac{d^2 w}{dr^2} = 0, \tag{17}$$

(7) is unchanged, and (10) takes the form

$$\nabla^4 w = \frac{2h}{D}\left[\frac{Z}{2h} + \frac{1}{r}\left(\frac{d\chi}{dr}\frac{d^2 w}{dr^2} + \frac{dw}{dr}\frac{d^2 \chi}{dr^2}\right)\right], \tag{18}$$

∇^2 having the significance stated in (16). The boundary conditions are

when $r = 0$: $$\frac{1}{r}\frac{d\chi}{dr} = \widehat{rr} = \widehat{\theta\theta} = \frac{d^2\chi}{dr^2}, \quad \frac{dw}{dr} = 0,$$

when $r = a$: $$0 = \frac{Eu}{r} = \widehat{\theta\theta} - \sigma\,.\,\widehat{rr} = \frac{d^2\chi}{dr^2} - \frac{\sigma}{r}\frac{d\chi}{dr}, \quad \frac{dw}{dr} = 0,$$

i.e.

$$\left.\begin{aligned} &\frac{dw}{dr} = 0 \text{ when } r = 0 \text{ and when } r = a,\\ &r\frac{d}{dr}\left(\frac{1}{r}\frac{d\chi}{dr}\right) = 0 \text{ when } r = 0, \quad r\frac{d}{dr}\left(\frac{1}{r}\frac{d\chi}{dr}\right) + \frac{1-\sigma}{r}\frac{d\chi}{dr} = 0 \text{ when } r = a.\end{aligned}\right\} \tag{19}$$

8. If now we write

$$x \text{ for } r^2, \text{ so that } 2r\,dr = dx, \quad \phi \text{ for } r\frac{dw}{dr} = 2x\frac{dw}{dx}, \quad \psi \text{ for } r\frac{d\chi}{dr} = 2x\frac{d\chi}{dx}, \tag{20}$$

equations (17) and (18) transform respectively into

$$\frac{d}{dx}\left[8x\frac{d^2\psi}{dx^2}+\frac{E}{x}\phi^2\right] = 0 \quad \text{and} \quad \frac{d}{dx}\left[4x\frac{d^2\phi}{dx^2}-\frac{h}{D}\left\{\frac{Zx}{2h}+\frac{2}{x}\phi\,.\,\psi\right\}\right] = 0. \tag{21}$$

Physical considerations require χ, w, and their differentials of all orders to be continuous at the centre ($x = 0$) and to be even functions of r, thus showing ϕ and ψ to be of order r^2 ($= x$) *at least*. The particular boundary conditions (19) of our problem require that

$$\left.\begin{aligned} \phi/\surd x\left(=\frac{dw}{dr}\right) &= 0 \text{ when } x = 0 \quad \text{and} \quad \text{when } x = a^2,\\ \frac{d\psi}{dx} = \frac{\psi}{x} \text{ when } x = 0, \quad &= \frac{1+\sigma}{2}\frac{\psi}{x} \text{ when } x = a^2. \end{aligned}\right\} \tag{22}$$

9. Since $4x\dfrac{d^2\psi}{dx^2} = r\dfrac{d}{dr}\nabla^2\chi = 0$ at the centre of the plate, the conditions at the centre make the constant of integration zero as regards the first of (21), and a similar argument leads to a like conclusion in regard to the second. Accordingly we may replace (21) by

$$\frac{d^2\psi}{dx^2}+\frac{E}{8}\frac{\phi^2}{x^2} = 0, \quad \frac{d^2\phi}{dx^2}-\frac{h}{2D}\left\{\frac{Z}{4h}+\frac{\phi\,.\,\psi}{x^2}\right\} = 0, \tag{23}$$

the factor x being cancellable in view of the symmetry of χ and w. From (23), when terms of the second order in ϕ and ψ are neglected, we have in virtue of (22)

$$\left.\begin{aligned} &\psi = 0, \text{ i.e. } \chi = \text{const., by (20)},\\ \phi = \frac{Z}{16D}x(x-a^2), \quad \text{whence} \quad w &= \frac{Z}{64D}(x^2-2a^2x+\text{const.}), \text{ by (20)},\\ &= \frac{Z}{64}(a^2-r^2)^2, \end{aligned}\right\} \tag{24}$$

since w is required to vanish at the edge ($r = a$). These are known results.

10. Allowance for the second-order terms in (23) can be made by a process of continued approximation. Using ϕ_0, ψ_0 to denote the foregoing 'small-deflexion solution', we express the wanted 'large-deflexion solution' in the form

$$\phi = \phi_0+\phi', \quad \psi = \psi_0+\psi' \quad (\psi_0 = 0), \tag{25}$$

and we denote by $\phi_1, \phi_2, \ldots$ and by $\psi_1, \psi_2, \ldots$, etc., our successive approximations to ϕ' and ψ'. We derive these from the formulae

$$\frac{d^2}{dx^2}\psi_n = -\frac{E}{8x^2}(\phi_0+\phi_{n-1})^2, \quad \frac{d^2}{dx^2}\phi_n = \frac{h}{2Dx^2}(\phi_0+\phi_{n-1})\,\psi_n, \tag{26}$$

—which are easily deduced from (23)–(25),—giving n the values 1, 2, 3, ... in turn. Then of the functions on the right of equations (26), the first is known at the start of every stage, the second when the first of (26) has been solved to determine ψ_n.

It will be convenient to write

$$\left.\begin{aligned} &x'' \text{ for } \frac{x}{a^2}, \quad \psi'' \text{ for } \frac{h}{2D}\psi, \quad \phi'' \text{ for } \tfrac{1}{4}\phi\sqrt{(hE/D)}, \\ &\text{so that } \phi''_0 \text{ stands for} \quad \frac{Za^4}{64}\sqrt{\left(\frac{hE}{D^3}\right)}\,x''(x''-1). \end{aligned}\right\} \tag{27}$$

Then equations (26) reduce to

$$\frac{d^2}{dx''^2}\psi''_n + \frac{1}{x''^2}(\phi''_0+\phi''_{n-1})^2 = 0, \quad \frac{d^2}{dx''^2}\phi''_n = \frac{1}{x''^2}(\phi''_0+\phi''_{n-1})\,\psi''_n. \tag{28}$$

They can be treated by relaxation methods if differentials are replaced by their finite-difference approximations, viz.

$$\left(\frac{d\psi''}{dx''}\right)_{x''} \text{ by } \frac{1}{2h}(\psi''_{x''+h}-\psi''_{x''-h}), \quad \left(\frac{d^2\psi''}{dx''^2}\right)_{x''} \text{ by } \frac{1}{h^2}(\psi''_{x''+h}-2\psi''_{x''}+\psi''_{x''-h}), \tag{29}$$

and $d\phi''/dx''$, $d^2\phi''/dx''^2$ by similar expressions.

The forms of the boundary conditions (22) are not altered by the substitutions (27). Solutions to (28) and (22) must be sought on the basis of a definite value for the parameter

$$\frac{Za^4}{64}\sqrt{\left(\frac{hE}{D^3}\right)} = \mu \quad \text{(say)}, \tag{30}$$

which is a measure of the transverse loading Z, and which is easily shown to be 'non-dimensional'.

11. The formulation of 'residuals', and their 'liquidation' by a systematic imposition of 'displacements', are routine processes which do not call for detailed description here. Figure 2 (relating to the case $\mu = 1$) exemplifies the progress of the successive approximations (§10). The accepted curves for ϕ'' and ψ'' are those lettered A and B respectively: having these, we can complete the solution as below.

According to (20) and (27)

$$\left.\begin{aligned} w &= \int\frac{\phi}{2x}dx = 2\sqrt{\left(\frac{D}{hE}\right)}\int\frac{\phi''dx''}{x''}, \\ \text{therefore} \quad w/2h &= \sqrt{\frac{D}{Eh^3}}\int_1^{r^2/a^2}\frac{\phi''dx''}{x''} = \sqrt{\frac{D}{Eh^3}}\,w'' \quad \text{(say)} \end{aligned}\right\} \tag{31}$$

at radius r, when w is assumed to vanish at the line $(r = a)$. The factor

$$\sqrt{\frac{D}{Eh^3}} = \sqrt{\frac{2}{3(1-\sigma^2)}} = 0{\cdot}8559 \quad \text{when} \quad \sigma = 0{\cdot}3. \tag{32}$$

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Accepting curve *A*, figure 2, as sufficiently exact, we deduced the curve of ϕ''/x'' in figure 3, and from this, by integration, the variation of w'' according to (31) and (32).

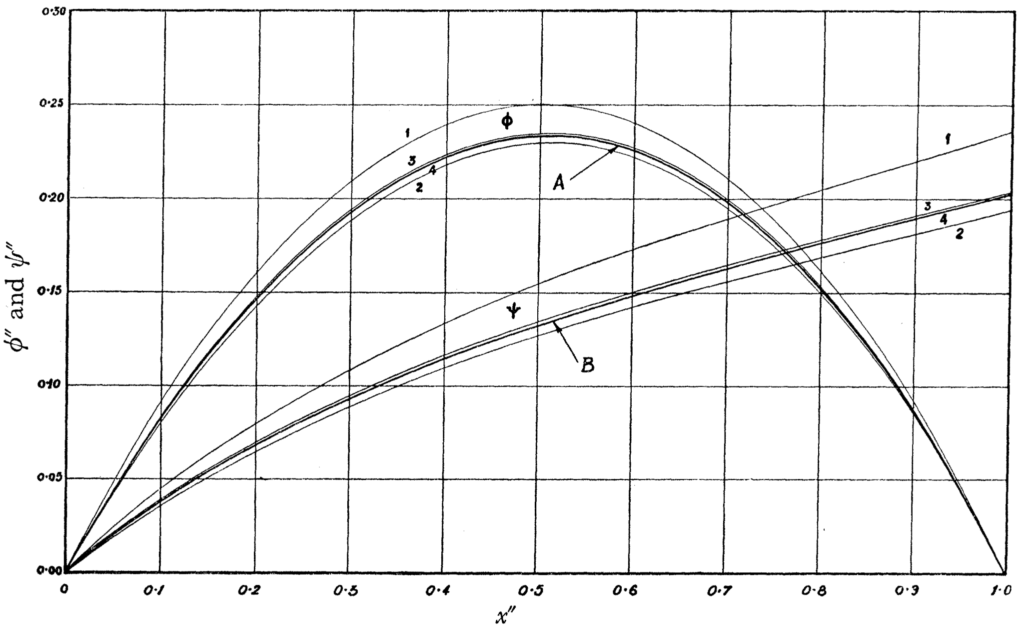

FIGURE 2

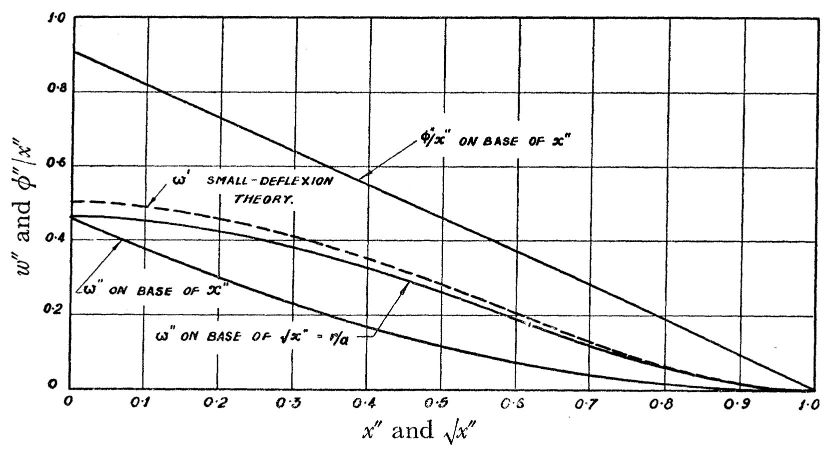

FIGURE 3

In figure 3, w'' is plotted both against x'' and against $\sqrt{x''}$ $(= r/a)$; also, for comparison, w'' as derived by ordinary (small deflexion) theory. For the central deflexion w_0 we have

$$w_0/2h = 0{\cdot}856 \times 0{\cdot}4614 = 0{\cdot}3949$$

in this instance, where $\mu = 1$ so that

$$\frac{Za^4}{E(2h)^4} = 4\mu\left(\frac{D}{Eh^3}\right)^{\frac{3}{2}} = 4 \times 0{\cdot}8559^3 = 2{\cdot}508 \qquad (33)$$

according to (30) and (32).

12. Other quantities evaluated by Way (§ 7) were

(I) the bending stress at the centre of the plate,

(II) the membrane stress at the centre of the plate,

(III) the membrane stress at the edge of the plate,

all expressed as multiples of $E(2h)^2/a^2$. In our notation

$$(\text{I}) = \frac{Eh}{1-\sigma^2}\left[\frac{1}{r}\frac{d\phi}{dr}-(1-\sigma)\frac{\phi}{r^2}\right]_{r=0} = \frac{Eh}{1-\sigma}\left(\frac{\phi}{x}\right)_{x=0} = 4\sqrt{\left(\frac{2}{3(1-\sigma^2)}\right)}\frac{Eh^2}{(1-\sigma)\,a^2}\left(\frac{\phi''}{x''}\right)_{x''=0},$$

$$\text{so bending stress (I)}\times\frac{a^2}{E(2h)^2} = \frac{1}{1-\sigma}\sqrt{\left(\frac{2}{3(1-\sigma^2)}\right)}\left(\frac{\phi''}{x''}\right)_{x''=0} = 1\cdot22\left(\frac{\phi''}{x''}\right)_{x''=0}; \tag{34}$$

$$(\text{II}) = \left(\frac{1}{r}\frac{d\chi}{dr}\right)_{r=0} = \left(\frac{\psi}{x}\right)_{x=0} = \frac{2D}{ha^2}\left(\frac{\psi''}{x''}\right)_{x''=0},$$

$$\text{so membrane stress (II)}\times\frac{a^2}{E(2h)^2} = \frac{1}{3(1-\sigma^2)}\left(\frac{\psi''}{x''}\right)_{x''=0}$$

$$= 0\cdot366_5\times[\text{slope of } \psi''-x'' \text{ curve at origin } (x''=0)]; \tag{35}$$

$$(\text{III}) = \left(\frac{1}{r}\frac{d\chi}{dr}\right)_{r=a} = \frac{2D}{ha^2}(\psi'')_{x''=1} = \frac{4}{3}\frac{E}{1-\sigma^2}\frac{h^2}{a^2}(\psi'')_{x''=1},$$

$$\text{so membrane stress (III)}\times\frac{a^2}{E(2h)^2} = \frac{1}{3(1-\sigma^2)}(\psi'')_{x''=1} = 0\cdot366_5(\psi'')_{x''=1}. \tag{36}$$

We have made corresponding calculations for $\mu = 1, 2, 3, 4, 5$, this range being more than sufficient to cover all cases having practical importance. Figure 4 (based on table 1) exhibits the agreement between our results and Way's.

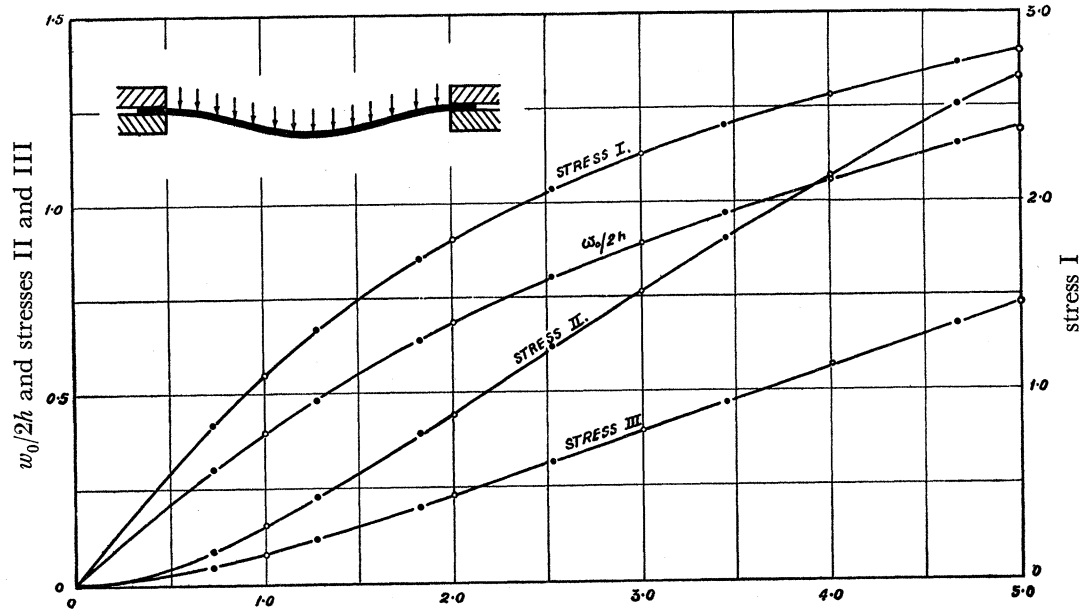

FIGURE 4. Way's result ●, our results ○.

TABLE 1. WAY'S RESULTS IN OUR NOTATION*

μ	0·724	1·275	1·818	2·520	3·443	4·67
$w_0/2h$	0·296	0·482	0·637	0·800	0·970	1·152
flexural stress I	0·834	1·327	1·704	2·069	2·408	2·727
membrane stress II	0·085	0·226	0·392	0·616	0·900	1·258
membrane stress III	0·041	0·112	0·196	0·311	0·468	0·671

13. The convergence of successive approximations, fairly rapid in figure 2 (i.e. for $\mu = 1$), was found to be slower for higher values of μ. This feature was expected, and to meet it certain modifications were made in the procedure outlined in § 10.

Given ϕ_{n-1}, we deduced from (26) the form of ψ_n and thence the form of ϕ_n. We took its value for some fixed value of x'' (usually 0·5) as a measure of ϕ, assuming that convergence of this measure would entail approximate convergence everywhere; and at the end of every cycle we plotted (figure 5) a point P_n of which the abscissa was the measure of ϕ_{n-1}, the ordinate was the measure of ϕ_n. On the assumption stated, if a curve be drawn through successive points of this kind, convergence ($\phi_n/\phi_{n-1} \to 1$) will be attained when the curve through $P_n, P_{n+1}, P_{n+2}, \ldots$ intersects the line $Y = X$ in figure 5, and the point of intersection will define the proper value which should be given to our measure of ϕ_n. Accordingly, having assumed an abscissa and deduced the ordinate of P_1, we took this ordinate as the abscissa and deduced the ordinate of P_2. Then, joining P_1P_2 by a straight line cutting the line $Y = X$ in Q, we took the abscissa of Q as our third estimate of ϕ, and so deduced P_3 in the same way. Finally, drawing a curve through $P_1P_3P_2$, we used this to deduce a point P_4 so close to the line $Y = X$ that further approximation was not necessary.

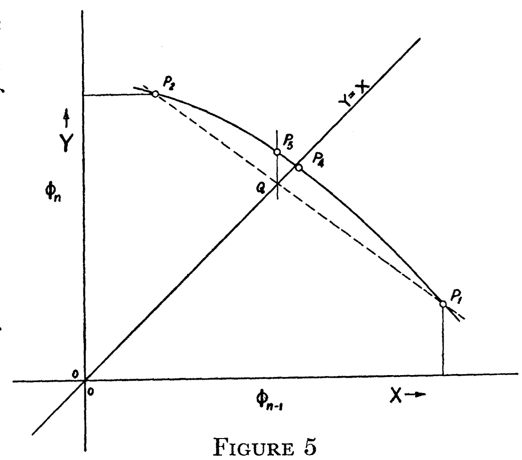

FIGURE 5

Figure 6 shows the development of a diagram of this kind in relation to the case where $\mu = 4$, figure 7 the convergence of the corresponding approximations to ϕ and ψ.

RELAXATION METHODS APPLIED TO SIMPLE EXAMPLES.

(2) AN EXAMPLE OF ELASTIC STABILITY

14. Friedrichs & Stoker (1941) have solved exactly (by orthodox methods) the case of a circular plate of radius a, having a 'simply supported' edge, which buckles with radial symmetry under the action of a uniformly distributed edge thrust. This problem too makes a convenient test case, specially simple in that only one independent variable (r) is involved.

* Our μ = Way's $qu_0^4/2{\cdot}508$ (cf. equation (33)).

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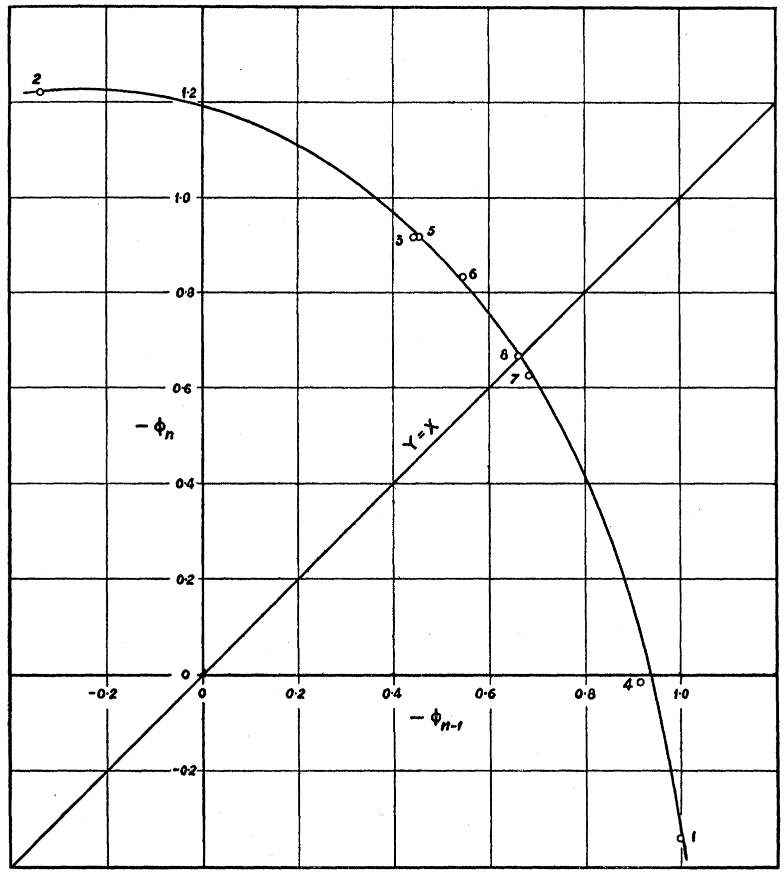

FIGURE 6

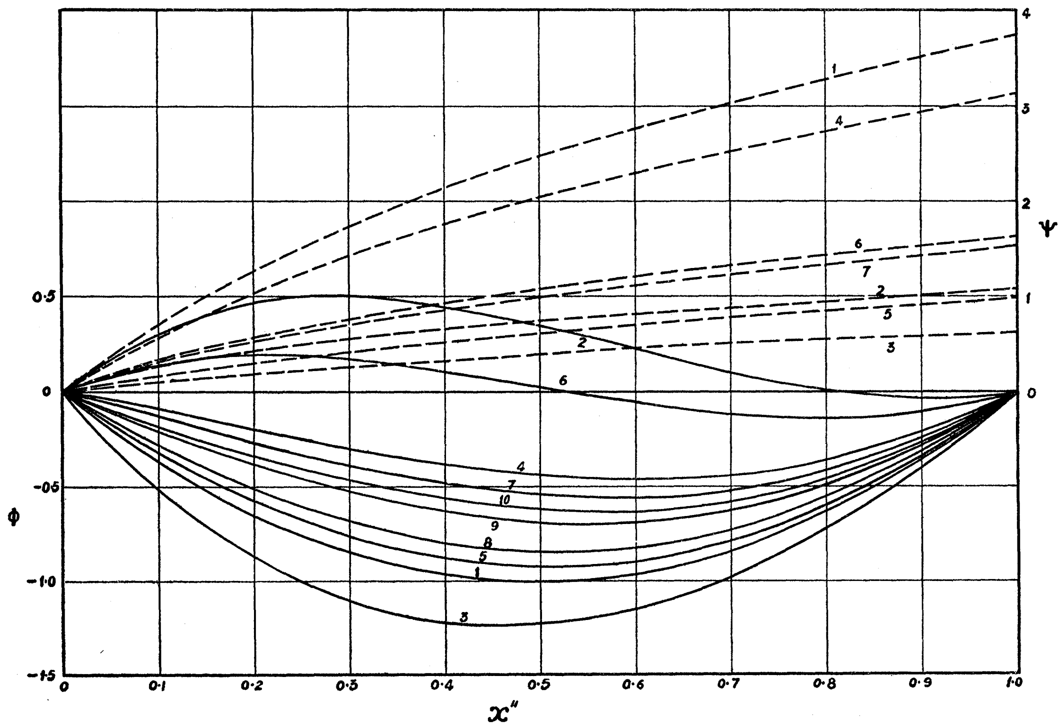

FIGURE 7

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Following Friedrichs & Stoker, we replace (6) and (15) by their equivalents in polar co-ordinates (r, θ), then, ignoring all differentials with respect to θ, we integrate the substituted equations to obtain *in our notation**

$$r\frac{d}{dr}\nabla^2\chi+\tfrac{1}{2}E\left(\frac{dw}{dr}\right)^2=0, \quad Dr\frac{d}{dr}\nabla^2 w=\frac{dw}{dr}\left(2h\frac{d\chi}{dr}-rP\right), \tag{37}$$

both constants of integration being zero in virtue of the conditions at the centre $(r=0)$. P denotes the line intensity of the applied edge thrust, which produces initially a uniform compressive stress $P/2h$ in all directions. χ defines (cf. § 6) the *additional* stresses resulting from the large deflexions.

$$\left.\begin{aligned}&\text{Now writing} && x \text{ for } r^2, \quad \phi \text{ for } r\frac{dw}{dr}, \quad \psi \text{ for } r\frac{d\chi}{dr},\\ &\text{as in § 8, and} && x'' \text{ for } \frac{x}{a^2}, \quad \psi'' \text{ for } \frac{h}{2D}\psi, \quad \phi'' \text{ for } \tfrac{1}{4}\phi\sqrt{(hE/D)},\end{aligned}\right\} \tag{38}$$

as in § 10, we reduce (37) to the 'non-dimensional' forms

$$\frac{d^2\psi''}{dx''^2}+\frac{\phi''^2}{x''^2}=0, \quad \frac{d^2\phi''}{dx''^2}=\frac{\phi''(\psi''-\lambda x'')}{x''^2}, \quad \text{where} \quad \lambda=a^2P/4D. \tag{39}$$

These are consistent with (8) and (9) of (F. & S. 1941), in which the edge thrust P is included in the stress function.

The purpose of our separation of P will be apparent from the account of method which follows. It means that χ must entail no radial stress at the edge $(r=a)$, where accordingly $\partial\chi/\partial r$ must vanish. On w, at the edge, the condition of simple support imposes the condition

$$\frac{d^2w}{dr^2}+\frac{\sigma}{r}\frac{dw}{dr}=0, \quad \text{i.e.} \quad \frac{d\phi}{dr}=(1-\sigma)\frac{\phi}{r},$$

so the edge conditions of our problem are

$$\psi''=0, \quad 2\frac{d\phi''}{dx''}=(1-\sigma)\frac{\phi''}{x''}, \quad \text{when} \quad x''=1. \tag{40}$$

Expected features of a 'large deflexion' solution

15. The special difficulty of 'buckling' problems such as this may be seen by considering the simplest case of a straight and uniform strut. There the neutral elastic stability indicated by a treatment on the basis of infinitesimal deflexions is known to be

* Hereafter we shall use (F. & S. 1941) as an abbreviated reference to the cited paper, in which ϕ and R replace χ and a as employed here, $-p$ is our ψ/r^2, and $-q$ is our $a\phi/r^2$.

§ 1 of the paper contains some misprints. In particular ϕ and w have been interchanged in its equations (4) and (5), (6) and (7); γ^2 is omitted from (7) and η^2 from (9); R^{-2} has been written for R^2 in the definition of its operator Γ.

followed by recovery of stability when the deflexions become large (cf., e.g., Southwell 1941, § 476): analogously we must expect that the characteristic number λ, in the second of (39), for small deflexions will have a 'critical value' independent of their amplitude, but for larger deflexions will rise with increasing amplitude, indicating recovery of stability. We must also (from strut theory) expect the mode to alter progressively as buckling develops. Below, both expectations are realized.

Estimation of the critical edge thrust. (*a*) By orthodox analysis

16. The 'critical value' of λ for small deflexions may be derived from (39) with the second-order terms suppressed, i.e. from

$$\frac{d^2\psi''}{dx''^2} = 0, \quad \frac{d^2\phi''}{dx''^2} + \frac{\lambda\phi''}{x''} = 0. \tag{41}$$

The first of these, with (40), shows that $\psi'' = 0$ everywhere. The second (cf., e.g., Forsyth 1914, § 111) is a form of Riccati's equation, integrable in the form

$$\phi'' = Ax''^{\frac{1}{2}} J_1\{2\sqrt{(\lambda x'')}\} \quad (A \text{ arbitrary}) \tag{42}$$

when (as here) ϕ'' is required to have a zero value for $x'' = 0$. Substituting in the second of (40), we find this boundary condition to require that

$$zJ_1'(z) + \sigma J_1(z) = 0, \quad \text{i.e.} \quad (1-\sigma)\, J_1(z) = zJ_0(z), \quad \text{when} \quad z = 2\sqrt{\lambda}. \tag{43}$$

From tables the first three roots of (43) are found to be (*when* $\sigma = 0{\cdot}318$, *the value assumed in F. & S.* 1941)

$$\left.\begin{array}{llll} & 2{\cdot}0600_2, & 5{\cdot}393, & 8{\cdot}574, \\ \text{giving as critical values of } \lambda & & & \\ \lambda = \tfrac{1}{4}z^2 = & 1{\cdot}0609_0, & 7{\cdot}271, & 18{\cdot}378. \end{array}\right\} \tag{44}$$

The gravest value gives $a^2P/D = 4\lambda = 4{\cdot}2436$ according to (39). This agrees with the value (4·24) given in F. & S. (1941, § 1).

Estimation of the critical edge thrust. (*b*) By relaxation methods

17. If the second of (41) had not been integrable, relaxation methods could have been applied to it on the basis of 'Rayleigh's principle', as in Parts VII B and VII C. From it we have

$$\lambda \int_0^1 \frac{\phi''^2}{x''}\, dx'' = -\int_0^1 \phi'' \frac{d^2\phi''}{dx''}\, dx'', \tag{45 A}$$

$$= \int_0^1 \left(\frac{d\phi''}{dx''}\right)^2 dx'' - \left[\phi'' \cdot \frac{d\phi''}{dx''}\right]_0^1,$$

$$= \int_0^1 \left(\frac{d\phi''}{dx''}\right)^2 dx'' - (1-\sigma)\left[\frac{\phi''^2}{2x''}\right]_0^1, \quad \text{when the second of (40) is satisfied,}$$

$$= \int_0^1 \left\{\left(\frac{d\phi''}{dx''}\right)^2 - (1-\sigma)\frac{d}{dx''}\left(\frac{\phi''^2}{2x''}\right)\right\} dx''. \tag{45 B}$$

The integral on the left is equivalent to

$$\frac{a^2hEP}{32D^2}\int_0^a r\left(\frac{dw}{dr}\right)^2 dr,$$

and the integral on the right of (45) B is equivalent to

$$\frac{a^2hE}{32D}\int_0^a r\left[(\nabla^2 w)^2 - \frac{2(1-\sigma)}{r}\frac{dw}{dr}\frac{d^2w}{dr^2}\right] dr,$$

so (45) A or B expresses the equality of (decrease in strain energy of thrust) to (increase in strain energy of flexure). The form (45) A is better suited to computation.

Working with the whole range $0 \leqslant x'' \leqslant 1$ divided into N equal intervals h, we may replace $d\phi''/dx''$, in the second of (40), and $d^2\phi''/dx''^2$, in (41) and in (45) A, by their finite difference approximations as under:

$$2h\left(\frac{d\phi''}{dx''}\right)_{x''} = (\phi'')_{x''+h} - (\phi'')_{x''-h}, \quad h^2\left(\frac{d''\phi''}{dx''^2}\right)_{x''} = (\phi'')_{x''-h} - 2(\phi'')_{x''} + (\phi'')_{x''+h}. \qquad (46)$$

We may also replace the integrals in (45) A by summations in accordance with Bickley's 'N-strip' formula for numerical integration, or by some other approximate formula (e.g. Simpson's rule), or graphically (i.e. by counting squares).

Then, for some *assumed* mode, we may deduce from (45) A a corresponding 'Rayleigh estimation' of λ, and use this estimate to compute 'residual forces' for each point of subdivision in the range from

$$\mathbf{F}_{x''} = \frac{\lambda}{x''}(\phi'')_{x''} + \frac{1}{h^2}\{(\phi'')_{x''-h} - 2(\phi'')_{x''} + (\phi'')_{x+h}\}, \qquad (47)$$

which is consistent with (41) and (46);* keeping λ constant we may effect a partial liquidation of the forces; then we may make a fresh estimate of λ from (45) A, and so on. The whole process has been given in previous papers (Parts VI, VII B, VII C).

Here we are concerned only with the gravest (i.e. lowest) critical value of λ, which we term λ_1. Exact treatment (§ 16) gives

$$\lambda_1 = 1{\cdot}0609_0: \qquad (44)\ bis$$

proceeding as above (from a starting assumption which satisfied (40) and made ϕ'' quadratic in x'')† we obtained the estimates

$$1{\cdot}06338, \quad 1{\cdot}062_5, \quad 1{\cdot}062_5, \quad 1{\cdot}062_5 \qquad (48)$$

in successive stages. This is satisfactory agreement (the error of the final estimate is $+0{\cdot}15\,\%$). Table 2 summarizes the computations.

* A different expression holds at the edge ($x'' = 1$), where $(\phi'')_{x''+h}$ relates to a 'fictitious point' and must be adjusted to satisfy the edge condition (40).

† This assumption gave an analytic expression for λ (not depending on approximate integration). Consequently more significant figures appear in the first of (48).

TABLE 2

	$x=0$	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0	1·1*	λ
Stage 1:													
ϕ_0'' (assumed quadratic form)	0	24,174	46,348	66,523	84,698	100,873	115,047	127,221	137,395	145,570	151,745	155,920	1·06338
$h^2 \dfrac{d^2\phi_0''}{dx''^2}$	0	−2,000	−1,999	−2,000	−2,000	−2,001	−2,000	−2,000	−1,999	−2,000	−2,000	—	—
$h^2\lambda_0\phi_0''/x''$	0	2,571	2,464	2,358	2,252	2,145	2,039	1,933	1,826	1,720	1,614	—	—
$\mathbf{F}_0 = h^2\left(\dfrac{\lambda_0\phi_0''}{x''}+\dfrac{d^2\phi''}{dx''^2}\right)$	0	571	465	358	252	144	39	−67	−173	−280	−386	—	—
Stage 2:													
ϕ_1'' (after liquidation of $\mathbf{F}_0$'s)	0	25,197	47,824	67,986	85,791	101,343	114,751	126,119	135,555	143,164	149,054	153,330	$1{\cdot}062_5$
$\mathbf{F}_1$	0	107	76	51	26	9	−8	−18	−27	−29	−31	—	—
Stage 3:													
ϕ_2'' (after liquidation of $\mathbf{F}_1$'s)	0	25,355	48,034	68,171	85,901	101,351	114,649	125,914	135,265	142,817	148,678	152,957	$1{\cdot}062_5$
$\mathbf{F}_2$	0	17	9	7	1	1	−3	−3	−3	−5	−3	—	—
Stage 4:													
ϕ_3'' (final mode)	0	25,371	48,049	68,177	85,890	101,322	114,601	125,850	135,189	142,731	148,587	152,864	$1{\cdot}062_5$
$\mathbf{F}_3$	0	2	2	−1	0	0	−1	0	−2	−1	−1	—	—

* Cf. footnote *, §17.

A generalization of Rayleigh's principle for large deflexions

18. Having a close estimate of λ_1 and of its associated mode (ϕ''_1, say), we may proceed to a corresponding (relaxation) treatment of the large-deflexion equations (39). So far the absolute magnitude of ϕ''_1 has been unrestricted: now, giving it some moderately large value at $x''=1$, we may satisfy the first of (39) and of (40) after replacing $d^2\psi''/dx''^2$, in the former equation, by its finite-difference approximation in the manner of (46). The resulting estimate of ψ'' can be inserted in the second of (39), from which we then derive a new estimate of ϕ''_1.

Again the estimate of λ_1 will alter with the form of ϕ''_1, but now (cf. § 15) λ must be expected to rise with increasing amplitude of deflexion, whereas in § 17 it was stationary. Moreover, in § 17 it was shown that (45) A expresses the constancy, as between the flat and a nearly flat configuration, of the total strain energy (extensional *plus* flexural): now, when $\lambda > \lambda_1$, the total strain energy of the *largely* bent configuration will be different from that of the flat, although both will be stationary for small variations. Accordingly 'Rayleigh's principle', the basis of our treatment in § 17, will not (as normally applied) serve for a treatment of large deflexions. We now consider whether any modification is feasible.

19. From the second of (39) we have

$$\lambda \int_0^1 \frac{\phi''^2}{x''}\,dx'' = \int_0^1 \frac{\phi''^2 \, . \, \psi''}{x''^2}\,dx'' - \int_0^1 \phi''\,\frac{d^2\phi''}{dx''^2}\,dx'',$$

$$= -\int_0^1 \left(\psi''\,\frac{d^2\psi''}{dx''^2} + \phi''\,\frac{d^2\phi''}{dx''^2}\right) dx'' \tag{49}$$

when the first of (39) is satisfied. Hence, for small variations $\delta\phi''$, $\delta\psi''$,

$$\delta\lambda \int_0^1 \frac{\phi''^2}{x''}\,dx'' + 2\lambda \int_0^1 \frac{\phi''\,.\,\delta\phi''}{x''}\,dx'' = -\int_0^1 \left(\delta\psi''\,\frac{d^2\psi''}{dx''^2} + \psi''\,\frac{d^2}{dx''^2}\,\delta\psi'' + \delta\phi''\,\frac{d^2\phi''}{dx''^2} + \phi''\,\frac{d^2}{dx''^2}\,\delta\phi''\right) dx'',$$

and the condition for a stationary value of λ (namely, $\delta\lambda = 0$) can be written as

$$2\int_0^1 \left\{\delta\phi''\left(\frac{\lambda\phi''}{x''} + \frac{d^2\phi''}{dx''^2}\right) + \delta\psi''\,\frac{d^2\psi''}{dx''^2}\right\} dx'' + \left[\phi''\,\frac{d}{dx''}\,\delta\phi'' - \frac{d\phi''}{dx''}\,\delta\phi'' + \psi''\,\frac{d}{dx''}\,\delta\psi'' - \frac{d\psi''}{dx''}\,\delta\psi''\right]_0^1 = 0 \tag{i}$$

for all permissible variations.

The terms in square brackets cancel at both limits since $\delta\phi''$, $\delta\psi''$, as well as ϕ'', ψ'', are subject to the boundary conditions (40). If then $\delta\psi''$ and $\delta\phi''$ are treated as *independent*, the conditions for a stationary value of λ are

$$\frac{d^2\psi''}{dx''^2} = 0, \quad \frac{\lambda\phi''}{x''} + \frac{d^2\phi''}{dx''^2} = 0, \tag{ii}$$

so are quite distinct from (39) unless ψ'' is negligible (as is the fact when the deflexions are infinitesimal); and if ψ'' and ϕ'' are related by the first of (39), so that

$$\frac{d^2}{dx''^2}\delta\psi''+\frac{2\phi''.\delta\phi''}{x''^2}=0, \tag{iii}$$

then (i) is equivalent to

$$\int_0^1 \delta\phi''\left(\frac{\lambda\phi''}{x''}+\frac{d^2\phi''}{dx''^2}-\frac{2\psi''\phi''}{x''^2}\right)dx''=0,$$

whence the conditions for a stationary value of λ are the first of (39) combined with

$$\frac{d^2\phi''}{dx''}+\lambda\frac{\phi''}{x''}=\frac{2\psi''\phi''}{x''^2}, \quad \text{everywhere.} \tag{iv}$$

Again we have not reproduced *both* of equations (39).

20. On the other hand, (iv) with the factor 2 on the right-hand side suppressed is identical with the second of (39), and it would be obtained in that form from (49) if the term $\psi''\frac{d^2\psi''}{dx''^2}$, in the latter equation, were halved. So the wanted function ϕ''_1 has the property that it gives a stationary value for μ as deduced from

$$\mu\int_0^1\frac{\phi''^2}{x''}dx''=-\int_0^1\left(\tfrac{1}{2}\psi''\frac{d^2\psi''}{dx''^2}+\phi''\frac{d^2\phi''}{dx''^2}\right)dx'', \tag{50}$$

when ψ'' and ϕ'' are related by the first of (39). But λ and μ as defined by (49) and (50) are not identical; the relation between them being

$$2(\lambda-\mu)\int_0^1\frac{\phi''^2}{x''}dx''=-\int_0^1\psi''.\frac{d^2\psi''}{dx''^2}dx'', \tag{51}$$

when ψ'' and ϕ'' are related by the first of (39).

With neglect of second-order terms we should have $\psi''=0$, therefore $\lambda=\mu$ according to (51). Then (50) is equivalent to

$$\lambda\int_0^1\frac{\phi''^2}{x''}dx''=-\int_0^1\phi''\frac{d^2\phi''}{dx''^2}dx'', \tag{45) A \textit{bis}$$

which was the equation used in § 17 to determine λ_1.

21. Arguing on the lines of Rayleigh's principle we may say that μ, being stationary in the required configuration, will be insensitive to small variations of the mode: therefore to a first approximation we may calculate μ (and proceed to deduce λ)* without allowance for the difference between the wanted mode ϕ'' and the mode (ϕ''_1, say) which corresponds with λ_1. Proceeding on this basis we have *approximately*, from (49),

$$\lambda\int_0^1\frac{\phi_1''^2}{x''}dx''=-\int_0^1\psi''_1\frac{d^2\psi''_1}{dx''^2}dx''-\int_0^1\phi''_1\frac{d^2\phi''_1}{dx''^2}dx'',$$

* This further step, of course, is not strictly warranted by the argument.

ψ_1'' being related with ϕ_1'' in accordance with the first of (39); and *exactly*, from (45) A of § 17,

$$\lambda_1 \int_0^1 \frac{\phi_1''^2}{x''} dx'' = -\int_0^1 \phi_1'' \frac{d^2\phi_1''}{dx''^2} dx''.$$

Therefore *approximately*

$$\left.\begin{aligned} \lambda &= \lambda_1 - \int_0^1 \psi_1'' \frac{d^2\psi_1''}{dx''^2} dx'' \bigg/ \int_0^1 \frac{\phi_1''^2}{x''} dx'', \\ &= \lambda_1 + \Delta\lambda \quad \text{(say)}, \end{aligned}\right\} \tag{52}$$

where $\Delta\lambda$ *is a positive quantity proportional to the square of the amplitude of* ϕ_1''. In other words, the relation contemplated in § 15 as holding between λ, the characteristic number, and A, the amplitude of the deflexion, may be expected to have *approximately* the form

$$\lambda = \lambda_1 + kA^2, \quad k \text{ being constant and calculable.} \tag{53}$$

Moreover it may be expected that the error of this approximation (as in the usual statement of Rayleigh's principle) is on the side of excess, i.e. that λ is *overestimated* by (53).*

In the present instance ϕ_1'' has been determined in table 2 (§ 17). Deducing ψ'' from the first of (39), *and defining A as the value of ϕ_1'' for $x'' = 1$*, we obtained for the constant k in (53) the value $0{\cdot}394_5$. Accordingly our extension of Rayleigh's principle leads to the assertion that

$$\lambda < \lambda_1 + kA^2 = 1{\cdot}062_5 + 0{\cdot}394_5 A^2. \tag{54}$$

22. Having this result we attempted to go further and impose a lower limit on λ in the manner of Southwell 1941, §§ 518–20. The wanted mode, since it makes μ as deduced from (50) a minimum, must differ from both of the modes which yield minimum values of μ_1, μ_2 as defined by

$$\mu_1 \int_0^1 \frac{\phi''^2}{x''} dx'' = -\int_0^1 \phi'' \frac{d^2\phi''}{dx''^2} dx'', \quad 2\mu_2 \int_0^1 \frac{\phi''^2}{x''} dx'' = -\int_0^1 \psi'' \frac{d^2\psi''}{dx''^2} dx'', \tag{55}$$

when ψ'' and ϕ'' are related by the first of (39). Therefore if (μ_1), (μ_2) denote the minimum values of μ_1, μ_2, we have

$$\mu > (\mu_1) + (\mu_2), \tag{56}$$

in which $(\mu_1) = \lambda_1$, as may be seen by comparing the first of (55) with (45) A, which defined λ_1 in § 17. Hence, according to (51),

$$\lambda > \lambda_1 + 2(\mu_2), \tag{57}$$

where (μ_2) stands for the minimum value of μ_2 as defined by the second of (55), and accordingly is proportional to A^2.

* This again is not a strict deduction from the argument. It is probable, because we are concerned with the smallest value of λ.

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But this result has no value for the reason that $(\mu_2) = 0$. In virtue of the first of (39), the variations of ψ'' and ϕ'' are related by

$$\frac{d^2}{dx''^2}\,\delta\psi'' + \frac{2\phi''}{x''^2}\,\delta\phi'' = 0,$$

and on that understanding an application of the Calculus of Variations to the second of (55) shows that when μ_2 has a stationary value then

$$\phi''(\mu_2 x'' - \psi'') = 0, \quad \text{everywhere.}$$

Hence, either ϕ'' must be zero everywhere (a nugatory result) or $d^2\psi''/dx''^2$ must be zero everywhere, and then $\mu_2 = 0$. Since $\mu_2 \nless 0$ according to the second of (55), this is its stationary value.

Even if we impose the condition

$$\phi'' = 1 \quad \text{when} \quad x'' = 1,$$

we can still make μ_2 zero by assuming that $\psi'' \propto x''$ excepting over an infinitesimal range of x'' in the neighbourhood of $x'' = 1$. So $(\mu_2) = 0$ in (57), which accordingly shows merely that $\lambda > \lambda_1$: this was already evident from (45) A.

23. However, even without the desired lower limit practical value attaches to the result of § 21, since it is obvious that *an inequality of the type of* (54) *can be deduced in relation to any example of elastic instability, once the critical loading* (λ_1) *has been determined, from one additional calculation which entails no more than the evaluation of a definite integral.* Further computations will of course be necessary when the mode, as well as λ, is wanted with close accuracy. In that event we may revert to the treatment outlined in § 18, to which only one remark need be added:—In any stage, given an estimate of ϕ'' we can deduce the corresponding estimate of ψ'': it will save time if at this point, i.e. *before* proceeding to a new estimate of ϕ'' based on the second of (39), a value of λ for insertion in that equation is deduced from (49).

We start with the advantages, now, of a known upper limit to λ *and of the knowledge that* μ *is stationary in the required configuration*: consequently, if we proceed on the basis of moderate increments to the value of A, the range of our explorations will not be wide. Thus when A is small we shall not be far off the mark if we take ϕ_1'' (§ 21) as our starting approximation to ϕ'', and λ as given by the right-hand side of (54). Similarly, having determined the mode and value of λ (ϕ_2'' and λ_2, say) which are appropriate to this small value of A, we can narrow the range of exploration when A is doubled; for it is clear from (39) that the wanted mode will differ from ϕ_2'', and accordingly the wanted value of λ will be overestimated if we deduce it from (52) with ϕ_1'' replaced by ϕ_2'' and ψ_1'' replaced by ψ_2'' as related with ϕ_2'' by the first of (39). *Knowing* ϕ_2'' *we can impose an upper limit lower than what is given by* (54).

Results for large deflexions in Example 2

24. Figure 8 exemplifies these remarks in relation to calculations in which σ was assumed to have the value 0·318 (§ 16). The critical value of λ (corresponding with $A = 0$) was determined in table 2, § 17, and the parabolic curve A represents the upper

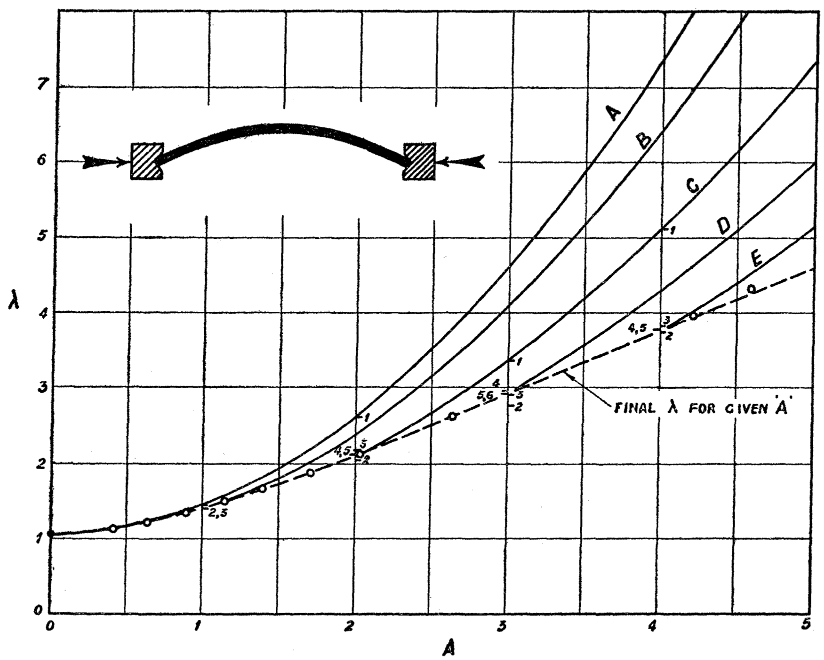

FIGURE 8. Open circles give results of (F. & S. 1941).

limit fixed by (54) of § 21. Computations summarized in table 3 (and started on the assumption $\phi''_{A=1} \equiv \phi''_{A=0}$: cf. § 23) yielded an accepted estimate

$$\lambda_{A=1} = 1{\cdot}4025, \tag{58}$$

and showed that

$$\lambda_{A>1} < 1{\cdot}0680 + 0{\cdot}3345A^2, \tag{59}$$

i.e. that λ lies below the parabolic curve B of figure 8. When $A > 1$ this gives an upper limit lower (and therefore closer) than that afforded by curve A. Subsequent and similar calculations gave the additional parabolas C, D, E of figure 8, based on the inequalities

$$\left.\begin{aligned} &(\text{curve } C)\quad \lambda_{A>2} < 1{\cdot}1194 + 0{\cdot}2491A^2,\\ &(\text{curve } D)\quad \lambda_{A>3} < 1{\cdot}2198 + 0{\cdot}1900A^2,\\ &(\text{curve } E)\quad \lambda_{A>4} < 1{\cdot}3463 + 0{\cdot}1518A^2. \end{aligned}\right\} \tag{60}$$

25. The convergence of the computations typified by table 3 is indicated by the numbered points in figure 8. Thus in relation to $A = 2$ we started on the basis of the

TABLE 3*

	$x = 0$	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0	1·1
Stage 1:												
$10^4\phi$ (assumed mode)	0	1,697	3,215	4,565	5,754	6,792	7,688	8,449	9,083	9,597	10,000	10,297
$10^5\mathbf{F}_\psi = 10^3\phi^2/x^2$	0	2,879	2,584	2,315	2,069	1,845	1,642	1,457	1,289	1,137	1,000	—
$10^4\psi$ (by liquidation of $\mathbf{F}_\psi$'s)	0	991	1,694	2,138	2,351	2,357	2,179	1,836	1,348	731	0	—
estimated λ	1·441,2											
Stage 2:												
$10^4\mathbf{F}_\phi = 100\phi(\lambda/x - \psi/x^2)$	0	76	95	111	123	132	138	142	144	145	144	—
$10^4\phi$ (by liquidation of $\mathbf{F}_\phi$'s)	0	1,529	2,981	4,338	5,584	6,707	7,699	8,552	9,263	9,830	10,251	10,529
$10^4\phi$ (reduced to make $\phi = 1$ at $x = 1$)	0	1,491	2,908	4,231	5,447	6,543	7,510	8,342	9,036	9,589	10,000	10,271
$10^5\mathbf{F}_\psi = 10^3\phi^2/x^2$	0	2,224	2,114	1,989	1,854	1,712	1,566	1,420	1,276	1,135	1,000	—
$10^4\psi$ (by liquidation of $\mathbf{F}_\psi$'s)	0	847	1,472	1,886	2,101	2,130	1,988	1,690	1,250	681	0	—
estimated λ	1·400,9											
Stage 3:												
$10^4\mathbf{F}_\phi = 100\phi(\lambda/x - \psi/x^2)$	0	82	97	109	119	128	134	138	141	141	140	—
$10^4\phi$ (by liquidation of $\mathbf{F}_\phi$'s)	0	1,499	2,916	4,237	5,448	6,540	7,505	8,336	9,028	9,580	9,991	10,262
$10^4\phi$ (reduced to make $\phi = 1$ at $x = 1$)	0	1,501	2,919	4,240	5,453	6,546	7,512	8,343	9,036	9,589	10,000	10,271
$10^5\mathbf{F}_\psi = 10^3\phi^2/x^2$	0	2,252	2,130	1,998	1,858	1,714	1,567	1,421	1,276	1,135	1,000	—
$10^4\psi$ (by liquidation of $\mathbf{F}_\psi$'s)	0	852	1,479	1,893	2,108	2,136	1,993	1,694	1,252	683	0	—
estimated λ	1·402,5											

* Dashes have been suppressed; i.e. ϕ, ψ here replace ϕ'', ψ''.

accepted solution for $A = 0$, i.e. with a value of λ calculated from the right-hand side of (54) and so higher than what would be given by the right-hand side of (58). This overestimate, corrected in the manner of § 18, led to successive estimates which are numbered 1, 2, 3, 4, 5. The convergence is oscillatory, but the last two estimates are nearly identical.

Next, *on the basis of this solution for* $A = 2$, the case $A = 4$ was started. Curve C was thus available, and in figure 8 the point for $A = 4$ numbered 1 gives a starting estimate based on the first of (59). (A closer starting estimate could have been based on curve D if this had been available.) Again the convergence is oscillatory but rapid. The case $A = 3$ was in fact treated last, though it might have preceded the case $A = 4$.

26. In figure 8 the broken-line curve ('final λ for given A') is compared with 'exact' results, taken from (F. & S. 1941),* which are shown by open circles. The agreement throughout the range (which more than covers all cases having practical reality) is as close as could be desired.

Table 4 records the co-ordinates of the open circles in figure 8.

TABLE 4. RESULTS OF (F. & S. 1941), IN OUR NOTATION†

A	0	0·423	0·646	0·88	1·13	1·39	1·70	2·03	2·64	4·22	4·59
λ	1·0609	1·130	1·215	1·34	1·48	1·67	1·89	2·13	2·64	3·99	4·33

RELAXATION METHODS APPLIED TO HARDER EXAMPLES.

(3) AN EXAMPLE OF STATIC EQUILIBRIUM

27. Having established the accuracy of our methods as applied to these two test examples, we can proceed with some confidence to similar treatment of problems in which orthodox methods would entail great if not prohibitive labour.

Large deflexions resulting from uniform pressure acting on a rectangular plate with clamped edges have been studied by Way (1938). His approximate method, which entails an assumed form of distortion involving eleven parameters, and derives their values from equations based on energy considerations, has been summarized by Timoshenko (1940, § 71). Three forms of rectangle were examined, including the square: we now treat that case for comparison, first with the admittedly approximate results of Way, and secondly with the results of a more recent and elaborate study by Levy (1942). It is an unfortunate circumstance that σ was assumed by Way to have the value 0·3, by Levy the value 0·316 ($= \sqrt{0 \cdot 1}$).

* Cf. § 14 and footnote.

† Our A = (F. & S. 1941) $\dfrac{A \times q_e}{\sqrt{(p_e A/E)}} \times 0{\cdot}3641$. Our λ = (F. & S. 1941) $\Lambda \times 1{\cdot}0609$.

The governing equations

28. Here, of the alternative equations (6) and (11), the latter are to be preferred as making u and v the unknowns. The stresses in the middle surface are related with u, v, w by (3) of § 2, whereby for (10) we can substitute the equivalent relation

$$\nabla^4 w = \frac{Z}{D} + \frac{3}{h^2}\left[\frac{\partial^2 w}{\partial x^2}\left\{\frac{\partial u}{\partial x} + \sigma\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \tfrac{1}{2}\sigma\left(\frac{\partial w}{\partial y}\right)^2\right\} + \frac{\partial^2 w}{\partial y^2}\left\{\frac{\partial v}{\partial y} + \sigma\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 + \tfrac{1}{2}\sigma\left(\frac{\partial w}{\partial x}\right)^2\right\}\right.$$
$$\left. + (1-\sigma)\frac{\partial^2 w}{\partial x\,\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)\right]. \tag{61}$$

This, and

$$\left.\begin{aligned}\frac{\partial \Delta}{\partial x} + \frac{1-\sigma}{1+\sigma}\nabla^2 u + \frac{1}{2}\frac{\partial}{\partial x}\left\{\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right\} + \frac{1-\sigma}{1+\sigma}\frac{\partial w}{\partial x}\nabla^2 w = 0,\\ \frac{\partial \Delta}{\partial y} + \frac{1-\sigma}{1+\sigma}\nabla^2 v + \frac{1}{2}\frac{\partial}{\partial y}\left\{\left(\frac{\partial w}{\partial x}\right) + \left(\frac{\partial w}{\partial y}\right)^2\right\} + \frac{1-\sigma}{1+\sigma}\frac{\partial w}{\partial y}\nabla^2 w = 0,\end{aligned}\right\} \tag{11 bis}$$

are our governing equations. With Way we shall assume that

$$\sigma = 0{\cdot}3, \quad \text{so that} \quad \frac{1-\sigma}{1+\sigma} = \frac{7}{13}. \tag{62}$$

29. Writing $\quad x',\ y' = (x, y)/L, \quad u',\ v' = (u, v)\,L/h^2, \quad w' = w/h, \qquad (63)$

where $2h$ as before denotes the thickness of the plate and L is some representative dimension of its plan form, *here taken as the length of one side of the square*, we shall leave the forms of (61) and (11) unchanged except that

$$\frac{ZL^4}{hD} = \tfrac{3}{2}(1-\sigma^2)\frac{ZL^4}{Eh^4} = \alpha \text{ (say) replaces } \frac{Z}{D}, \quad 3 \text{ replaces } \frac{3}{h^2}. \tag{64}$$

Since the alterations required to make the governing equations 'non-dimensional' are so slight, we need not actually rewrite these in terms of x', y', u', v', w': instead, *we shall assume in what follows that x, y, u, v, w carry non-dimensional significance.* Solutions must be sought on the basis of particular values assumed for the numerical parameter α. Each will then apply to a whole family of geometrically similar plates, bent by pressures proportional to hD/L^4.

Energy relations

30. Equations (61) and (11), being conditions of equilibrium, could have been derived by variational methods as conditions for a stationary value of the total potential energy $\mathfrak{V}$. Modified as in § 29, they are conditions which must be satisfied in order that

$$\frac{L^2}{h^2}\frac{\mathfrak{V}}{D} = \frac{1}{2}\iint(\nabla^2 w)^2\,dx\,dy + \frac{3}{2}\iint\left(e_{xx}^2 + e_{yy}^2 + 2\sigma e_{xx}e_{yy} + \frac{1-\sigma}{2}e_{xy}^2\right)dx\,dy - \alpha\iint w\,dx\,dy \tag{65}$$

may be stationary for all permissible variations δu, δv, δw, when e_{xx}, e_{yy}, e_{xy} have the expressions (3) in terms of our 'non-dimensional' u, v and w. This last assertion is easily verified,* (11) being derived from (65) by variation of u and v, (61) by variation of w. On the right of (65), the first term relates to the strain energy of flexure, the second to the strain energy of extension, and the third to the potential of the (uniform) applied pressure.

31. Plainly, in Example 3 the equilibrium is stable, so the stationary value of $\mathfrak{V}$ is in fact a minimum; and the argument from energy (leading to a treatment entailing surface integrals) will occasionally be of value, used in conjunction with the normal technique of liquidation effected on a 'relaxation net'. For example, having distributions of u, v, w which constitute an approximation to the correct *type* of deflexion, we may want to decide their optimal *magnitudes*. These may be deduced from (65) as follows.

Multiplying the given values of w by k, according to (3) we shall leave the *distributions* of e_{xx}, e_{yy}, e_{xy} unaltered if at the same time we multiply u and v by k^2. Thereby we shall multiply the first integral on the right of (65) by k^2, the second by k^4, and the third by k: consequently $\mathfrak{V}$ *so far as it depends on* k will be given by an expression of the form

$$\frac{L^2}{h^2}\frac{\mathfrak{V}}{D} = k^2 I_1 + k^4 I_2 - \alpha k I_3, \tag{66}$$

where

$$\left.\begin{aligned} I_1 &= \frac{1}{2}\iint (\nabla^2 w)^2\,dx\,dy, \\ I_2 &= \frac{3}{2}\iint\left\{e_{xx}^2 + e_{yy}^2 + 2\sigma e_{xx}e_{yy} + \frac{1-\sigma}{2}e_{xy}^2\right\}dx\,dy, \\ I_3 &= \iint w\,dx\,dy \end{aligned}\right\} \tag{67}$$

have known (computed) values; so the condition for a stationary value of $\mathfrak{V}$ $\left(\text{namely } \dfrac{\partial\mathfrak{V}}{\partial k} = 0\right)$ is

$$2kI_1 + 4k^3 I_2 - \alpha I_3 = 0, \tag{68}$$

and from this the wanted value of k may be determined.

* For example, when u alone is varied in (65), then

$$\frac{L^2}{h^2}\frac{\delta\mathfrak{V}}{D} = 3\iint\left\{(e_{xx}+\sigma e_{yy})\,\delta\frac{\partial u}{\partial x} + \frac{1-\sigma}{2}\,e_{xy}\,\delta\frac{\partial u}{\partial y}\right\}dx\,dy, \quad \text{according to (3)},$$

$$= -3\iint \delta u\left\{\frac{\partial}{\partial x}(e_{xx}+\sigma e_{yy}) + \frac{1-\sigma}{2}\frac{\partial}{\partial y}e_{xy}\right\}dx\,dy, \quad \text{when } \delta u = 0 \text{ at the boundary.}$$

For stationary $\mathfrak{V}$, the cofactor of δu in the surface integral must vanish everywhere: hence, substituting from (3) for e_{xx}, ..., etc., we obtain the first of (11).

'Optimal synthesis'

32. Or again, having two computed distributions of u, v, w (say, u_A, v_A, w_A and u_B, v_B, w_B) which both satisfy the boundary conditions but which relate to different values (α_A, α_B, say) of the loading parameter α, we may want to deduce a good starting assumption for some other value of α.

Let k_A be the value of k (as obtained from (68) of § 31) which corresponds with α when u, v, w are assumed to have the distributions u_A, v_A, w_A; and let k_B relate similarly to u_B, v_B, w_B. Then $k_A w_A$ and $k_B w_B$ are alternative starting assumptions, of which the first may be expected to be the closer when α has a value near to α_A, the second when α has a value near to α_B; so in general it will be reasonable to assume that w is given by

$$(\alpha_A-\alpha_B)\,w = \alpha(k_A w_A - k_B w_B) + \alpha_A k_B w_B - \alpha_B k_A w_A. \tag{69}$$

Because equations (11) are not linear in w, the corresponding forms of u and v will not have similar expressions, but must be deduced from those equations (and from w) by computation *ad hoc*. Then, having starting assumptions for all of u, v, w, we can deduce an optimal multiplier k in the manner of § 31, thus defining completely the wanted starting assumption. Finally, the residual forces entailed by this assumption at nodal points can be calculated, and liquidated, on the chosen net.

The finite-difference equations

33. Details of computation are explained most simply by a worked example. First, in the 'non-dimensional' forms of (61) and (11) we substitute their finite-difference approximations for the integrals, then we solve the equations as thus modified by techniques which were described in Part VII A.

The operators involved in (61) and (11) are listed below, with their finite-difference approximations. The suffix numbering relates to figure 9.

$$\left.\begin{aligned}
&2a\left(\frac{\partial w}{\partial x}\right)_0 \approx w_1-w_3, \quad 2a\left(\frac{\partial w}{\partial y}\right)_0 \approx w_2-w_4,\\
&a^2\left(\frac{\partial^2 w}{\partial x^2}\right)_0 \approx w_1+w_3-2w_0, \quad a^2\left(\frac{\partial^2 w}{\partial y^2}\right)_0 \approx w_2+w_4-2w_0,\\
&4a^2\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)_0 \approx w_a-w_b+w_c-w_d, \quad a^2(\nabla^2 w)_0 \approx w_1+w_2+w_3+w_4-4w_0,\\
&a^4(\nabla^4 w)_0 \approx w_{\rm I}+w_{\rm II}+w_{\rm III}+w_{\rm IV}+2(w_a+w_b+w_c+w_d)\\
&\qquad\qquad -8(w_1+w_2+w_3+w_4)+20w_0.
\end{aligned}\right\} \tag{70}$$

Making the substitutions, we obtain in place of (61) an equation which may be written in the abbreviated form

$$a^4[\nabla^4 w] = a^4\alpha + 3\Phi(w_0, w_1, \ldots, \text{etc.}), \tag{71}$$

$a^4[\nabla^4 w]$ denoting the finite-difference approximation to $a^4\nabla^4 w$ as given in (70), and $\Phi(w_0, w_1, \ldots$, etc.) denoting a^4 multiplied by the finite-difference approximation, according to (70), to the expression in square brackets on the right of (61).

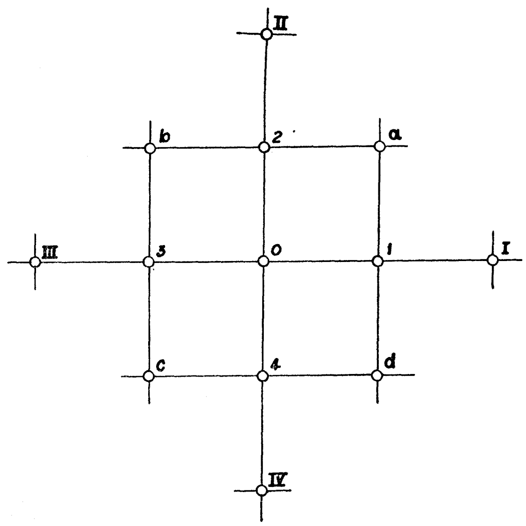

FIGURE 9

Outline of the relaxation attack

34. Small values of α will entail small values of w, and for these the term $\Phi(w_0, w_1, \ldots$, etc.) on the right of (71) can be suppressed. Then we have an equation of 'biharmonic' form, viz.

$$[\nabla^4 w] = \alpha. \tag{72}$$

This was solved by methods described in Part VII A, §§ 17–22, to obtain the mode for infinitesimal deflexions (figure 10).

Next, equations (11) *with these values of w inserted* were attacked in the manner of Part VII A to obtain a corresponding approximation to u and v, the boundary conditions being

$$u = v = 0, \quad \text{along every edge of the square plate.} \tag{73}$$

Both here and in succeeding determinations of u, v and w, the symmetry of the problem permitted attention to be concentrated on one-eighth of the complete square plate.

An obvious next step was to insert u, v and w *as thus computed* in the term $\Phi(w_0, w_1, \ldots$, etc.) on the right of (71), thereby converting that equation to the type of (72); to derive from it a new approximation to w; then to correct u, v; and so on by an iterative process. Results so obtained were found, however, to oscillate widely until the device of § 31 was employed to improve the trial solution by means of a calculated multiplier k. Then, residual forces as computed from*

$$\left.\begin{array}{c}\mathbf{F} = ka^4[\nabla^4 w] - a^4\alpha - 3k^3\mathfrak{F},\\ \text{in which } \mathfrak{F} \text{ denotes the value of } \Phi(w_0, w_1, \ldots, \text{etc.}) \text{ when } k = 1,\end{array}\right\} \tag{74}$$

* When **F** is zero everywhere, multiplication of (74) by w, and integration of the resulting equation over the whole area of the plate, lead (as they should) to (68).

were left *on the whole* nearer to zero than before. Relaxing them with a use of the standard biharmonic 'pattern' (i.e. on the basis that w is variable only in the first term on the right of (74)), we could improve the *distribution* of w (which is not affected by k) to form the basis of a new cycle of operations. In this way the iterative process was made more rapidly convergent.

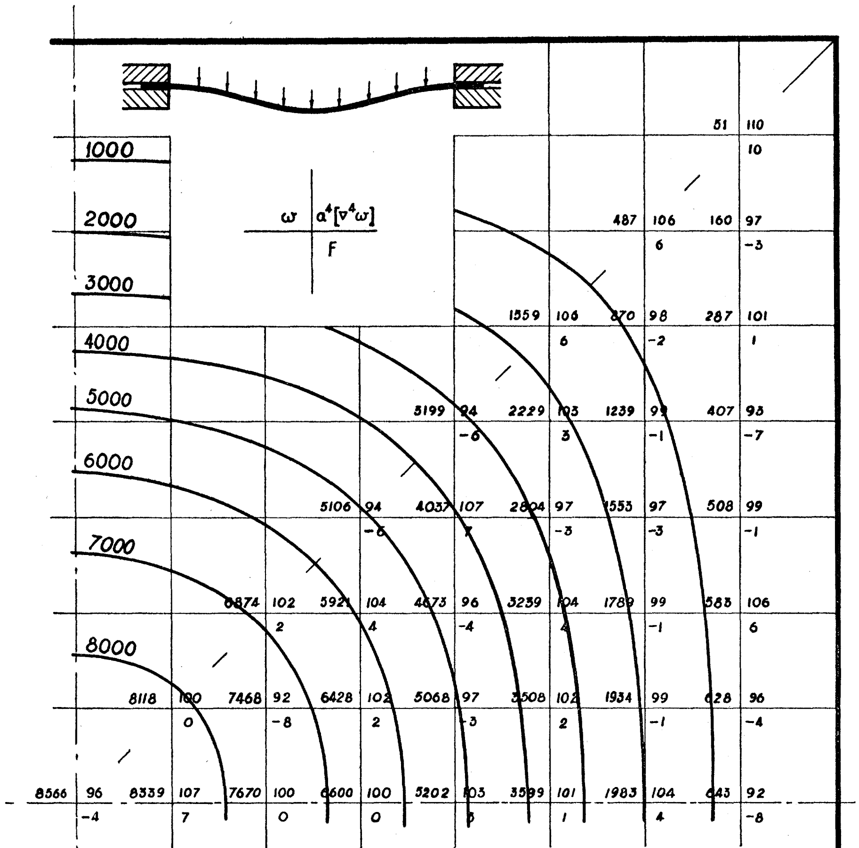

FIGURE 10. Mode for infinitesimal deflexions.

35. Proceeding in this manner, on a net of mesh-side $a = \frac{1}{8}$ we reached in three cycles a sufficiently exact solution for the case in which α, as defined in (64), had the value 4096 ($= 8^4$).

At this point advance was made to a net of mesh-side $\frac{1}{16}$. It is customary in such advance (Part III, § 13) to start from interpolated values of w, but here it was found better to interpolate values of $\nabla^4 w$ and, after some 'smoothing', derive the corresponding w's by relaxation with a use of the standard biharmonic 'pattern'. The resulting values of F as given by (74) were small, and a relatively short liquidation process sufficed to complete the solution for the finer net (figure 11).

36. In the next case studied, α was given a value roughly twice what had been taken previously, namely, 8736.* Since two solutions were available, viz. (1) the 'small

* This value was selected arbitrarily. For $\sigma = 0{\cdot}3$ it gives a value 400 to the parameter pa^4/Eh^4 of Levy (cf. § 27).

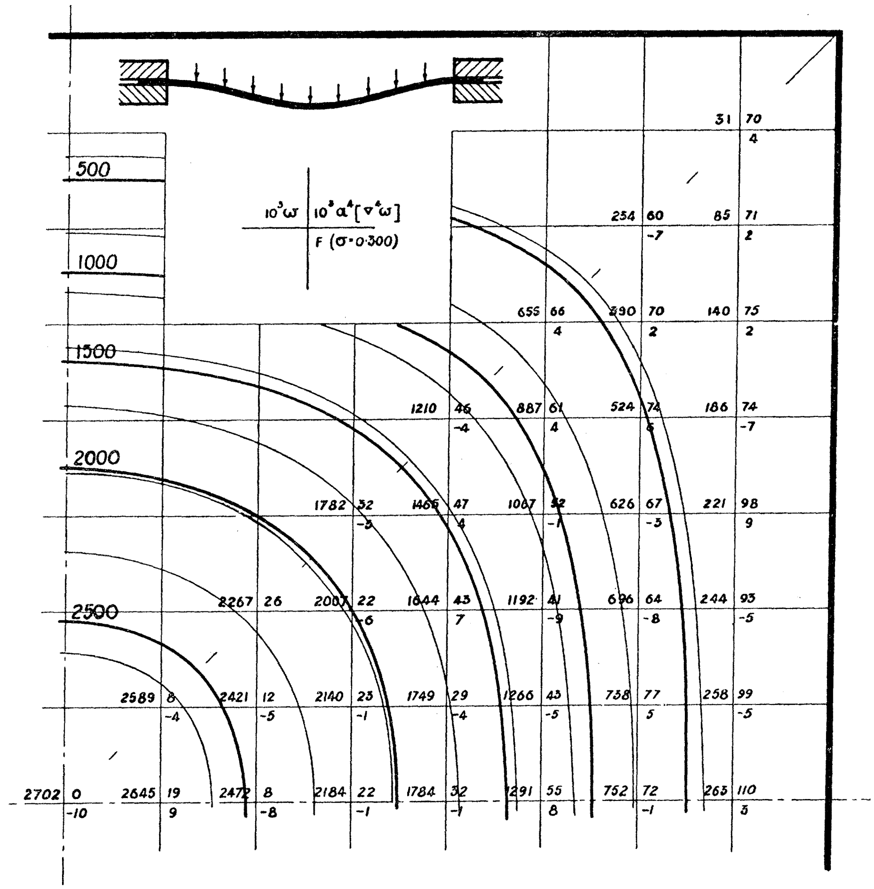

FIGURE 11. Mode for $\alpha = 8^4$. Load per mesh point $= 10^3 a^4 \alpha = 62\cdot5$.

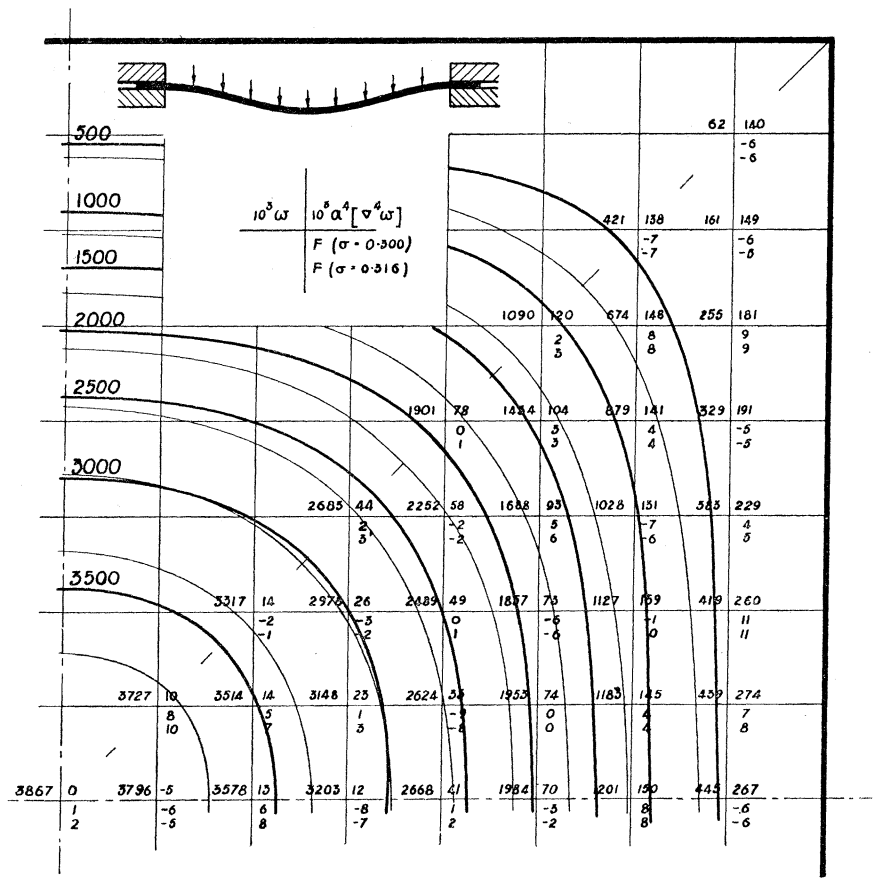

FIGURE 12. Mode for $\alpha = 8736$. Load per mesh point $= 10^3 a^4 \alpha = 133\cdot3$.

deflexion solution' of § 34 and figure 10, (2) the solution for $\alpha = 4096$ (figure 11), it was decided to seek a good starting solution for this new case by 'optimal synthesis' (§ 32). Thereafter the operations of § 34 were employed, and three cycles (two on a net of mesh-side $a = \frac{1}{16}$) led to the solution recorded in figure 12.

37. In figures 11 and 12, deflexions are shown to left of the nodal points, and on their right, in the top line values of $\alpha^4[\nabla^4 w]$—the load sustained by the flexural stresses —and in the bottom line values of the forces left unliquidated (**F**). To avoid decimals, all numbers have been multiplied by 1000. Contours of w are drawn in bold lines, and fine lines give, for comparison, contours for the mode of infinitesimal deflexion (reproduced from figure 10). Every contour has symmetry with respect both to medians and to diagonals of the square plate.

The contours in figure 10 have, of course, only relative significance, since α is there restricted only by the assumption that it is small. Consequently only general comparison can be made between the bold and fine-line contours in figures 11 and 12. But this will serve to show that the mode alters slowly with increasing pressure; and the point is made here to controvert a recent suggestion (Dunn 1942) that modal similarity may be assumed to imply a corresponding similarity in the ratio of flexural and 'membrane' resistance. Study of figures 11 and 12 reveals, notwithstanding their similarity in respect of w, that in figure 12 the flexural stresses attain more than proportionately large values close to the clamped edge.

38. Figure 13 reveals a satisfactory measure of agreement between our results and those of Way and Levy, over a range of loading more than sufficient to cover practical

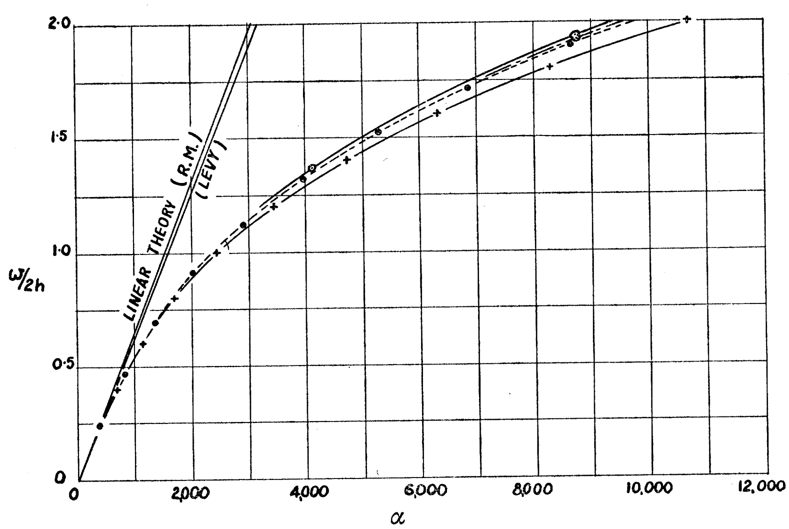

FIGURE 13. ● Levy, $\sigma = 0{\cdot}316$; + Way, $\sigma = 0{\cdot}300$; ⊙ R.M., $\sigma = 0{\cdot}300$ (left), $\sigma = 0{\cdot}316$ (right).

cases. In our computations, variation of σ between 0·3 and 0·316 (§ 27) had little effect, the lower value entailing a slightly larger deflexion. This is contrary to what would be concluded from a comparison of Way's results with Levy's; but it seems unlikely that all of the difference between their results can be explained by their assumption of different values for σ.

Relaxation methods applied to harder examples.

(4) An example of elastic stability

39. As a harder example of the type of Example 2 (§§ 14–26) we take the case of a square plate which buckles under shearing actions applied to bars whereby every edge is clamped and compelled to move as a rigid body. The critical loading (corresponding with *infinitesimal* deflexions) has been calculated by Iguchi (1938). We begin by computing this quantity on the basis of relaxation methods (cf. § 17).

The governing equation is (12) of § 5, with

$$P_x = P_y = 0, \quad S = \text{const.} = \mathbf{S}_c \quad (\text{say}), \tag{75}$$

i.e. it is

$$D\nabla^4 w - 2\mathbf{S}_c \frac{\partial^2 w}{\partial x \, \partial y} = 0, \tag{76}$$

$\mathbf{S}_c$ being the critical value of S. Multiplying through by w, then integrating over the whole plate with use of the boundary conditions

$$w = \frac{\partial w}{\partial \nu} = 0, \tag{77}$$

we obtain the relation

$$D\iint (\nabla^2 w)^2 \, dx \, dy = 2\mathbf{S}_c \iint w \frac{\partial^2 w}{\partial x \, \partial y} \, dx \, dy, \tag{78}$$

which may be used in accordance with Rayleigh's principle to determine $\mathbf{S}_c$.

Energy aspects of the equation governing small deflexions

40. Equation (78) can be interpreted in terms of energy. I_1 and I_2 as defined by

$$\left.\begin{aligned} I_1 &= \tfrac{1}{2}D\iint (\nabla^2 w)^2 \, dx \, dy \left(= \tfrac{1}{2}D\iint w\nabla^4 w \, dx \, dy\right), \\ I_2 &= \frac{hE}{1-\sigma^2}\iint \left\{e_{xx}^2 + e_{yy}^2 + 2\sigma e_{xx} e_{yy} + \frac{1-\sigma}{2} e_{xy}^2\right\} dx \, dy, \end{aligned}\right\} \tag{79}$$

measure, respectively, the total strain energies of flexure and of extension, when e_{xx}, e_{yy}, e_{xy} stand for the *total* extensional strains. When, on the other hand, e_{xx}, e_{yy}, e_{xy} stand for *additional* strains superposed on a uniform initial shear strain given by

$$\boldsymbol{e}_{xy} = \mathbf{S}/2\mu h = (1+\sigma)\,\mathbf{S}/Eh, \tag{80}$$

then in the expression for the strain energy of extension we must replace e_{xy} by $(\mathbf{e}_{xy}+e_{xy})$, thus obtaining a new expression in place of I_2, viz.

$$I_2' = I_2 + \frac{1+\sigma}{2}\frac{\mathbf{S}^2}{hE}\iint dx\,dy + \mathbf{S}\iint e_{xy}\,dx\,dy, \tag{81}$$

in which I_2 has its expression (79). Now clearly, in (81),

$$\frac{1+\sigma}{2}\frac{\mathbf{S}^2}{hE}\iint dx\,dy = \mathbf{U}_0, \tag{82}$$

—the total strain energy in the flat configuration of uniform shear strain; so the total strain energy (of both kinds) in a configuration defined by *additional* displacements u, v, w is given by

$$\mathbf{U} = I_1 + I_2' = I_1 + I_2 + \mathbf{U}_0 + \mathbf{S}\iint e_{xy}\,dx\,dy.$$

That is to say (since no work is done by the external forces of additional displacements *which vanish at the boundary*), the gain in total potential energy (of the bent as distinct from the flat configuration) is

$$\mathfrak{V}\ (\text{say}) = \mathbf{U} - \mathbf{U}_0 = I_1 + I_2 + \mathbf{S}\,.\,I_3, \tag{83}$$

where

$$I_3 = \iint e_{xy}\,dx\,dy = \iint\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)dx\,dy, \quad \text{according to (3)},$$

$$= \iint \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\,dx\,dy, \quad \text{since } u = v = 0 \text{ on the boundary},$$

$$= -\iint w\frac{\partial^2 w}{\partial x\,\partial y}\,dx\,dy. \tag{84}$$

41. When w is infinitesimal, u and v as determined from (11) are infinitesimals of the second order; therefore, if we neglect terms of higher order in w than the second, $I_2 = 0$ in (83) and equation (78), § 39, is equivalent to the statement that

$$\mathfrak{V} = 0, \quad \text{when} \quad \mathbf{S} = \mathbf{S}_c. \tag{85}$$

In other words, *when the critical loading is attained, not only has the total potential energy* $\mathfrak{V}$ *a stationary value in the flat configuration, but that value is not altered as a result of small displacements of the wanted type.* This (cf. Southwell 1941, § 492) amounts to the assertion that the stability of the flat configuration is neutral.

Rayleigh's principle makes the further assertion that $\mathbf{S}_c$ as deduced from (78) has a stationary value in the wanted mode: i.e. that

$$\delta I_1 + \mathbf{S}_c\,.\,\delta I_3 = 0 \tag{86}$$

for all permissible variations δw^* from this mode. It can be used to establish the governing equation (76): for since

$$\delta I_1 = D\iint \nabla^2 w \,.\, \delta\nabla^2 w \, dx\, dy = D\iint \delta w \,.\, \nabla^4 w \, dx\, dy, \quad \text{in virtue of (77)},$$

and since
$$\delta I_3 = -\iint\left(\delta w \frac{\partial^2 w}{\partial x\,\partial y} + w \,.\, \delta \frac{\partial^2 w}{\partial x\,\partial y}\right) dx\, dy$$

$$= -2\iint \delta w \frac{\partial^2 w}{\partial x\,\partial y}\, dx\, dy, \quad \text{in virtue of (77)},$$

equation (86) requires that

$$\iint \delta w \left(D\nabla^4 w - 2\mathbf{S}_c \frac{\partial^2 w}{\partial x\,\partial y}\right) dx\, dy = 0 \quad \text{for all permissible variations } \delta w,$$

therefore (76) is satisfied.

Energy aspects of the large-deflexion equations

42. When the distortion is considerable, I_2 can no longer be neglected in (83), and we must expect that $\mathbf{S}$ will exceed $\mathbf{S}_c$; consequently we can no longer expect $\mathfrak{V}$ to vanish as in (85), but on the other hand it must still (by a general theorem in Mechanics) have a stationary value in the wanted (equilibrium) configuration. That is to say, we may expect to obtain the governing equations as conditions which must be satisfied in order that

$$\delta\mathfrak{V} = \delta I_1 + \delta I_2 + \mathbf{S} \,.\, \delta I_3 = 0 \tag{87}$$

for all permissible variations δu, δv, δw.

Both of these anticipations are realized. For on varying u we find that

$$\delta I_2 = \frac{-2hE}{1-\sigma^2}\iint \delta u \left\{\frac{\partial}{\partial x}(e_{xx} + \sigma e_{yy}) + \frac{1-\sigma}{2}\frac{\partial}{\partial y} e_{xy}\right\} dx\, dy,$$

as in the footnote to § 30, while $\delta I_1 = \delta I_3 = 0$: consequently (87) leads to the first of (11). On varying v we arrive, similarly, at the second of (11); and on varying w (which enters into all of I_1, I_2, I_3) we arrive at

$$D\nabla^4 w = 2\mathbf{S}\frac{\partial^2 w}{\partial x\,\partial y} + \frac{2hE}{1-\sigma^2}\left[\frac{\partial}{\partial x}\left\{(e_{xx} + \sigma e_{yy})\frac{\partial w}{\partial x} + \frac{1-\sigma}{2} e_{xy}\frac{\partial w}{\partial y}\right\}\right.$$
$$\left. + \frac{\partial}{\partial y}\left\{(e_{yy} + \sigma e_{xx})\frac{\partial w}{\partial y} + \frac{1-\sigma}{2} e_{xy}\frac{\partial w}{\partial x}\right\}\right], \tag{88}$$

which is the form assumed in this example by (15) of § 6, the last of the governing equations.

* Neither I_1 nor I_3 varies with u or v.

But $\mathfrak{V} \neq 0$ when the governing equations are satisfied. For on multiplying (88) throughout by $\frac{1}{2}w$, then integrating over the whole plate with use of the boundary conditions (77) and of the three governing equations, we find that

whence, and by (83), we have
$$\left.\begin{aligned} I_1+2I_2+\mathbf{S}I_3 &= 0,\\ \mathfrak{V}+I_2 &= 0. \end{aligned}\right\} \tag{89}$$

I_2 being necessarily positive, $\mathfrak{V}$ according to (89) is negative: that is to say, the total potential energy is less in the wanted configuration than it is so long as the plate, loaded by a shearing action $\mathbf{S}$ in excess of $\mathbf{S}_c$, remains flat. This result too was to be expected (cf. § 18).

43. We have seen that the governing equations, and hence the wanted (equilibrium) configuration, can be deduced from (87), which in § 42 was obtained from a general theorem in Mechanics. But (87) also expresses the statement that $\mathbf{S}$ as deduced from

$$I_1+I_2+\mathbf{S}I_3 = 0, \tag{90}$$

is stationary for all permissible variations; *so the wanted configuration can be deduced from a principle akin to Rayleigh's.*

It is not the fact that satisfaction of the governing equations implies the satisfaction of (90),—i.e., according to (83), the evanescence of $\mathfrak{V}$: we have seen that $\mathfrak{V}<0$ in the wanted configuration. This circumstance, however, does not invalidate the use in computation of the principle stated in italics. Exactly similar conclusions were reached (§ 20) in relation to Example 2.

44. Following our treatment of that example, we may say that in the wanted configuration μ as deduced from

$$I_1+I_2+\mu I_3 = 0 \tag{91}$$

is stationary, and that
$$\mathbf{S} = \mu - I_2/I_3, \tag{92}$$

this last equation being a consequence of (91) and the first of (89). Equation (50), § 20, assumes the form (91), and equation (51) the form (92), when $\mathbf{S}$ is substituted for λ and

$$I_1 \text{ for } \int_0^1 \phi'' \frac{d^2\phi''}{dx''^2}\,dx'' = \frac{1}{32}\frac{hEa^2}{D}\int_0^a r\frac{dw}{dr}\frac{d}{dr}\nabla^2 w\,dr,$$

$$I_2 \text{ for } \frac{1}{2}\int_0^1 \psi'' \frac{d^2\psi''}{dx''^2}\,dx'' = \frac{h^2a^2}{16D^2}\int_0^a r\frac{d\chi}{dr}\frac{d}{dr}\nabla^2\chi\,dr,$$

$$I_3 \text{ for } \int_0^1 \frac{\phi''^2}{x}\,dx'' = \frac{hE}{8D}\int_0^a r\left(\frac{dw}{dr}\right)^2 dr;$$

and it is easy to verify that on this understanding, and when multiplied by $-32\pi D^2/hEa^2$, I_1 denotes the total strain energy of flexure, I_2 the total strain energy of extension corresponding with χ (§ 14), λI_3 the change in extensional strain energy which results from P—a quantity of the type of the term $\mathbf{S}\iint e_{xy}\,dx\,dy$ in (81). Consequently our treatment of Example 2 is basically identical with what has been given here.

45. As in § 21, arguing on the lines of Rayleigh's principle we may say that μ, being stationary in the required configuration, will be insensitive to small variations of the mode. Therefore to a first approximation we may calculate μ (and proceed to deduce $\mathbf{S}$) without allowance for differences between the wanted mode w and a mode already determined (w_1, say) which corresponds with some other value of $\mathbf{S}$.

'Non-dimensional' approximations in finite differences

46. Computations must of necessity be performed on the basis of finite-difference approximations to the governing equations, and in 'non-dimensional' variables. Substituting from (63), § 29, we have in place of (76)

$$\nabla^4 w - 2\lambda_c \frac{\partial^2 w}{\partial x\,\partial y} = 0, \tag{76 A}$$

and in place of (88)

$$\nabla^4 w - 2\lambda \frac{\partial^2 w}{\partial x\,\partial y} - 3\left[\frac{\partial}{\partial x}\left\{(e_{xx}+\sigma e_{yy})\frac{\partial w}{\partial x} + \frac{1-\sigma}{2}e_{xy}\frac{\partial w}{\partial y}\right\}\right.$$
$$\left. + \frac{\partial}{\partial y}\left\{(e_{yy}+\sigma e_{xx})\frac{\partial w}{\partial y} + \frac{1-\sigma}{2}e_{xy}\frac{\partial w}{\partial x}\right\}\right] = 0, \tag{88 A}$$

to be used in conjunction with the relations (3), § 2, *u, v, w, x and y now having non-dimensional significance*, and λ being a numerical 'loading parameter' defined by

$$\lambda = \mathbf{S}L^2/D. \tag{93}$$

These are our non-dimensional governing equations, and the energy relations of §§ 40–45 must be modified in accordance. I_1, I_2 as defined in (79) are dimensional quantities obtained by application of a multiplying factor h^2D/L^2 to I_1, I_2 as defined by (67) *with non-dimensional significance for u, v, w, x and y*; and the dimensional quantity

$$\mathbf{S}\iint e_{xy}\,dx\,dy = \frac{h^2D}{L^2}\lambda I_3$$

when a like significance is attached to all the symbols in (84). Consequently (83) can be replaced by

$$L^2\mathfrak{V}/h^2D = I_1 + I_2 + \lambda I_3 \tag{83 A}$$

when I_1, I_2, I_3 have non-dimensional significance; (85) and (86) by

$$\mathfrak{V} = I_1 + \lambda_c I_3 = 0 \tag{85 A}$$

and by

$$\delta I_1 + \lambda\,\delta I_3 = 0, \tag{86 A}$$

respectively; and (87) by

$$\delta I_1 + \delta I_2 + \lambda \, . \, \delta I_3 = 0, \tag{87 A}$$

which leads to (88) A in the same way that (87), § 42, led to (88). Finally, the first of (89), § 42, can be replaced by

$$I_1 + 2I_2 + \lambda I_3 = 0, \tag{89 A}$$

and the conclusion of § 44 may be restated by saying that in the wanted configuration μ as deduced from

$$I_1 + I_2 + \mu I_3 = 0 \qquad (91)\ bis$$

is stationary, and that

$$\lambda = \mu - I_2/I_3 \tag{90 A}$$

when non-dimensional significance is attached to u, v, w, x and y in calculating I_1, I_2, I_3. The practical deduction is as stated in § 45, with λ substituted for **S**.

47. The finite-difference approximations (70), § 33, employed as before, lead to the replacement of (76) A, § 46, by

$$\mathbf{F}_0 = a^4[\nabla^4 w] - \tfrac{1}{2}a^2\lambda_c(w_a - w_b + w_c - w_d), \tag{94}$$

and of (88) A, § 46, by

$$\mathbf{F}_0 = a^4[\nabla^4 w] - \tfrac{1}{2}a^2\lambda(w_a - w_b + w_c - w_d) - 3\Phi(w_0, w_1, \ldots, \text{etc.}), \tag{95}$$

Φ having the same significance as in § 33, and $\mathbf{F}_0$ as before denoting the residual force at 0. The derivation of 'relaxation patterns' from a given expression for $\mathbf{F}_0$ has been explained in earlier papers of this series, so needs no description here.

The integrations required in an energy treatment (§§ 40–45) are replaced, correspondingly, by summations in accordance with approximate formulas. It was remarked in Part VII C (§ 9) that 'Simpson's rule' will often give results as good, or better, than those deduced by more elaborate treatment: here we observe that when the integrand has both zero value and zero gradient at either end of its range, still better results may come from *simple summation*.

48. Let 0, 1, 2, ..., $2N-1$, $2N$ be points of subdivision in the range AB, figure 14, and suppose the integrand y, *of which both the value and the slope is zero at A and B*, to be extrapolated 'by reflexion' to fictitious points C and D, just outside the range. Then by hypothesis

$$y_A = y_B = 0, \quad y_C = y_1, \quad y_D = y_{2N-1}, \tag{i}$$

so of the two areas shaded, by Simpson's rule,

$$\left.\begin{aligned} 6\int_A^1 y\,dx &= 3\int_C^1 y\,dx = h(y_C + 4y_A + y_1) = 2hy_1 \\ \text{and} \qquad 6\int_{2N-1}^B y\,dx &\ \text{(similarly)} = 2hy_{2N-1}, \\ \text{where } 2Nh &= \text{total range } AB. \end{aligned}\right\} \tag{ii}$$

Hence, applying Simpson's rule in the range $1-(2N-1)$, we have

$$3\int_A^B y\,dx = 2h\{y_1+y_3+\ldots+y_{2N-1}+2(y_2+y_4+\ldots+y_{2N-2})\}, \tag{iii}$$

and direct application of the rule in the range AB gives

$$3\int_A^B y\,dx = 2h\{y_2+y_4+\ldots+y_{2N-2}+2(y_1+y_3+\ldots+y_{2N-1})\}, \tag{iv}$$

since $y_A = y_B = 0$. Adding equations (iii) and (iv) we have

$$\int_A^B y\,dx = h(y_1+y_2+\ldots+y_{2N-2}+y_{2N-1}) = h\sum_A^B (y), \tag{96}$$

—again, because $y_A = y_B = 0$.

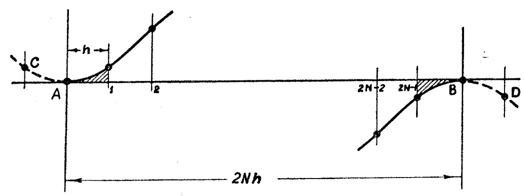

FIGURE 14

Outline of the relaxation attack

49. To solve (94), the equation governing *infinitesimal* deflexions, a mode was guessed and a corresponding estimate of λ deduced from (85) A, § 46; then, for that value of λ, residuals were deduced from (94) and liquidated with use of a relatively simple 'pattern'. This (figure 15) was deduced from (94) after giving λ a value correct to 2 figures. Greater accuracy would have entailed more labour, without compensating advantages.

A new estimate of λ could now be deduced from (85) A, and the foregoing operations repeated. Six cycles of this iterative process gave figure 16 as the mode for infinitesimal deflexions, and

$$\lambda_c = 150 \tag{97}$$

as the wanted (critical) value of the 'loading parameter' (§ 46).

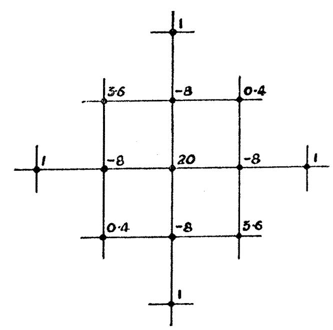

FIGURE 15

50. Figure 16 was made the starting assumption of an attack on the large-deflexion equations for (Case 1) a central deflexion equal to the plate thickness $2h$,—i.e., in the 'non-dimensional' notation of (63), § 29, for a central deflexion w_c having the value 2. λ having been estimated from (89) A, § 46, residuals were calculated from (95) and relaxed (almost completely) with a use of the standard 'biharmonic pattern' (cf. § 34);

then, the resulting deflexions were multiplied so as to restore w_c to its fixed value. In this way, starting from the mode of figure 16 (w_0, say), we arrived at a new mode termed w_1.

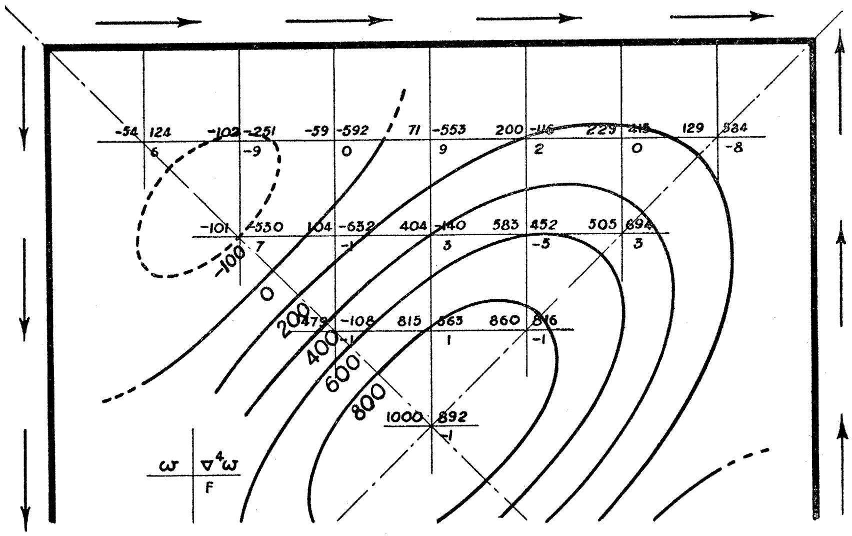

FIGURE 16. Mode for infinitesimal deflexions.

Next, with w_1 as starting assumption the whole process was repeated to give a new mode w_2 (say); then, the device of § 13 was employed to deduce a still closer approximation w_3. That is to say, for every nodal point a diagram of the type of figure 5 was constructed, P_1 and P_2 now having the co-ordinates (w_0, w_1) and (w_1, w_2), and Q giving by its (equal) co-ordinates a corresponding value for w_3. It was found that the residuals given by w_3 were much smaller than the initial values taken from w_0: they were relaxed with patterns deduced from (94)—i.e. with neglect of $\Phi(w_0, w_1, \ldots$, etc.) in (95)—but for a λ-value (196) computed from the large-deflexion relation (89) A, § 46.

Our purpose in thus using inexact 'relaxation patterns' was, of course, to avoid the complexity of a special pattern for every nodal point: those which we used were all of the type of figure 15, and differed only in that different numbers replaced the four there shown as 0·4 and 3·6. Notwithstanding this inexactitude of the relaxation process, continued improvement of the mode resulted because, at the start of every new cycle of operations, *residuals were calculated exactly from* (95). The accuracy of the final result (figure 17) is of course verifiable.

51. In Case 2, w_c was given the value 4 (central deflexion = twice plate-thickness). As a starting assumption, all deflexions were assumed to vary linearly with w_c in the range $0 \leqslant w_c \leqslant 4$, so that

$$(w)_{w_c=4} = 2(w)_{w_c=2} - (w)_{w_c=0}, \tag{98}$$

and the quantities on the right of (98) were taken from the previous solutions of §§ 49 and 50. The mode thus found was improved by relaxation effected with simplified patterns (cf. § 50).

Five cycles of such operations led to a solution deemed sufficiently exact (figure 18) and to a value 334 for λ. The results of §§ 49–51 are summarized in the table which follows:

TABLE 5

$\frac{1}{2}w_c = \frac{\text{central deflexion}}{\text{plate thickness}}$	0	1	2
λ	150	196	334
μ	150	174	248

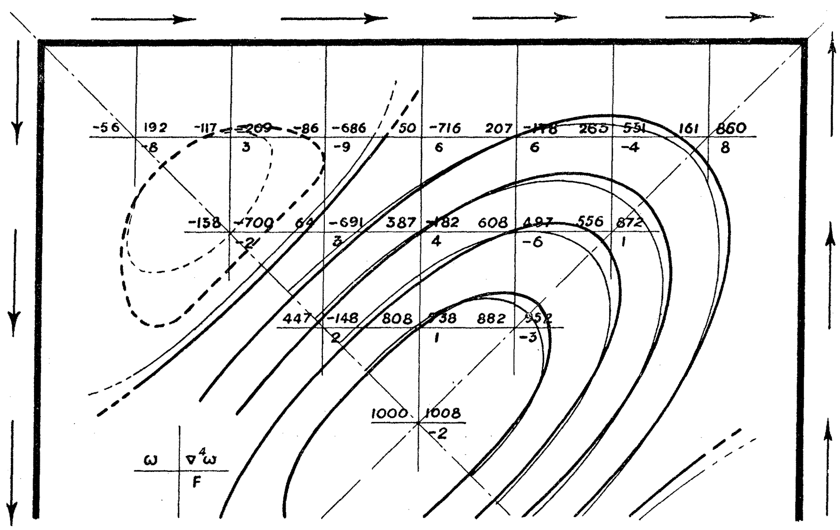

FIGURE 17. Case 1.

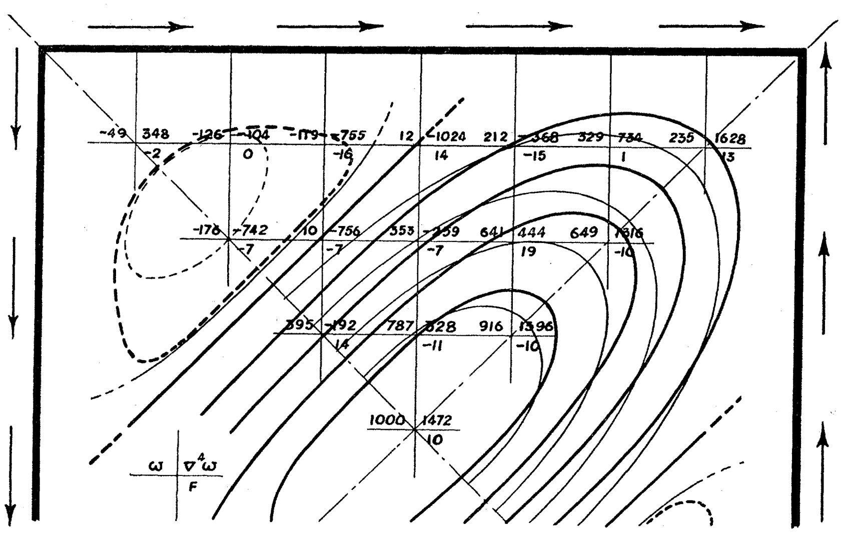

FIGURE 18. Case 2.

52. Figure 19, constructed from this table, serves to confirm the argument of § 45. The bold-line curve for λ is a parabola passing through the critical value $\lambda_c = 150$, and deduced from (89) A, § 46, on the assumption that w has the *distribution* found in § 49 (figure 16). For $w_c = 2$ (Case 1) the points numbered 1–4 record values of λ computed in successive cycles (§ 50). From the point 4 another parabola (shown in fine line) records the predictions drawn from (89) A, again on the assumption that the distribution is unchanged.

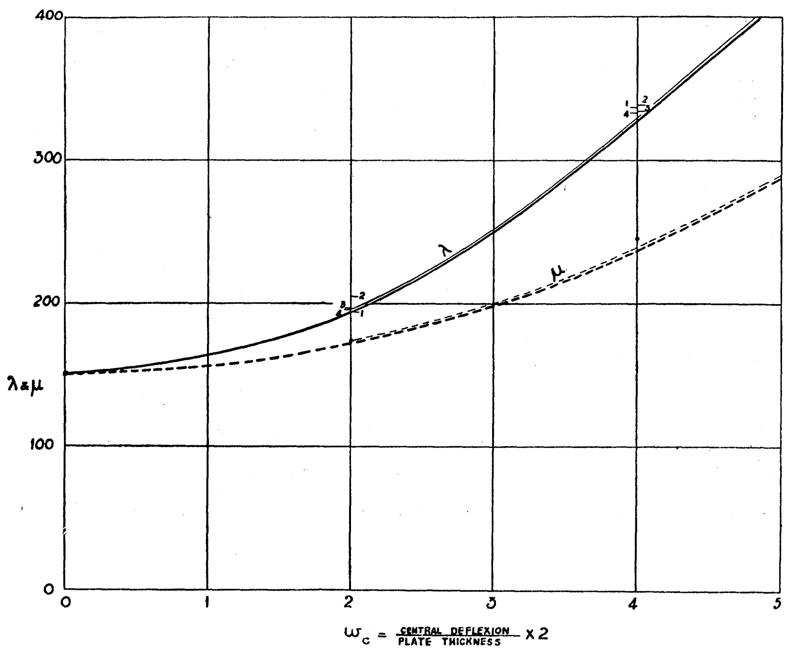

FIGURE 19

The broken-line curves for μ serve to illustrate the point that μ, in virtue of its stationary property (§ 45), is even more closely predicted by (91) than λ by (89) A, § 46. The bold-line parabola, passing through the critical value $\mu = 150$, gives within an accuracy of some 4 % *both* of the finally accepted values (shown by black dots).

CONCLUSION

53. Except in specially simple cases, the 'large-deflexion equations' of von Kármán (§ 2) have so far been found to present insuperable difficulties to orthodox analysis: treated by Relaxation Methods, they appear from this investigation to entail nothing worse than very considerable labour.

Our extension of Rayleigh's principle (§§ 18–22 and 42–5) provides a treatment likely to have considerable value in design. Just as, in such problems of elastic stability as the strength of struts, estimates of critical loadings are wanted but need not be exact, so in cases of 'well-developed buckling' (entailing 'tension fields' of the kind first brought to notice by Wagner) it will be important to have some early notion of the rate at which resistance is recovered as the deflexions increase.

References

Relaxation methods applied to engineering problems:

Part III. Christopherson, D. G. & Southwell, R. V. 1938 *Proc. Roy. Soc.* A, **168**, 317–350.

Part VI. Pellew, A. & Southwell, R. V. 1940 *Proc. Roy. Soc.* A, **175**, 262–290.

Part VII A. Fox, L. & Southwell, R. V. 1945*a* *Phil. Trans.* A, **239**, 419–460.

Part VII B. Christopherson, D. G., Fox, L., Green, J. R., Shaw, F. S. & Southwell, R. V. 1945*b* *Phil. Trans.* A, **239**, 461–487.

Part VII C. Allen, D. N. de G., Fox, L., Motz, H. & Southwell, R. V. 1945*c* *Phil. Trans.* A, **239**, 488–500.

Dunn, L. G. 1942 *Tech. Notes Nat. Adv. Comm. Aero., Wash.*, no. 859.

Forsyth, A. R. 1914 *A treatise on differential equations.* Macmillan and Co.

Friedrichs, K. O., & Stoker, J. J. 1941 *Amer. J. Appl. Mech.* **9**, A-7-14.

Iguchi, S. 1938 *Proc. Phys.-Math. Soc. Japan*, **20**, 814–832.

Kármán, Th. v. 1910 *Encykl. Math. Wiss.* **4** (iv), 349.

Levy, S. 1942 *Tech. Notes Nat. Adv. Comm. Aero., Wash.*, no. 847.

Southwell, R. V. 1941 *Introduction to the theory of elasticity*, 2nd ed. Oxford Univ. Press.

Timoshenko, S. P. 1940 *Theory of Plates and Shells.* McGraw Hill Co.

Way, S. 1934 *Trans. Amer. Soc. Mech. Engrs*, **56**, 627–636.

Way, S. 1938 *Proc. 5th Int. Congr. Appl. Mech. (Cambridge, Mass.)*, pp. 123–128. John Wiley and Sons.